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Potential Flow Through a Conical Pipe With an Application to Diffraction Theory

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Abstract

The primary objective of this report is to obtain an exact solution of the following potential problem: Find an axially symmetric solution of the potential equation having a vanishing normal derivative on the surface of a semi-infinite conical pipe with a circular aperture, and having a prescribed behavior at infinity. The solution may immediately be interpreted as the velocity potential of a steady-state irrotational flow of a non-viscous, incompressible fluid through a rigid conical pipe with a circular aperture. Using this interpretation, the solution is employed to derive, for suitable excitation, an approximate expression for the far field of the corresponding boundary-value problem involving the diffraction of sound.

The method is to construct integral representations of the solution using a variant of the Wiener-Hopf procedure. These representations lead to eigenfunction expansions of the potential from which exact expressions for the hydrodynamical conductivity of the opening are obtained. The behavior of the velocity of the fluid near the circular edge of the conical pipe is determined and is employed to prove the uniqueness of the solution. Using the Rayleigh static method, which requires the knowledge of the conductivity of the opening, we obtain the approximate far field 'outside' the pipe resulting from excitation 'inside' the pipe.

Table of Contents

1. Introduction

The primary objective of this report is to obtain an exact solution of the following potential problem: Find an axially symmetric solution of the potential equation having a vanishing normal derivative on the surface of a semi-infinite conical pipe with a circular aperture, and having a prescribed behavior at infinity. The solution may immediately be interpreted as the velocity potential of a steady-state irrotational flow of a non-viscous, incompressible fluid through a rigid conical pipe with a circular aperture. Using this interpretation, the solution is employed to derive, for suitable excitation, an approximate expression for the far fields of the corresponding boundary-value problem involving the diffraction of sound.

The method is to construct integral representations of the solution using a variant of the Wiener-Hopf procedure. These representations lead to eigenfunction expansions of the potential from which exact expressions for the hydrodynamical conductivity of the opening are obtained. The behavior of the velocity of the fluid near the circular edge of the conical pipe is determined and is employed to prove the uniqueness of the solution. For the corresponding radiation problem the approximate far field is obtained by assuming the wavelength of the sound waves to be large compared to the aperture dimensions and then using certain properties of the solution of the potential problem. In this connection two well-known procedures may be employed. Both are based on the fact that in the large wavelength limit the solution of the time-reduced wave equation is approximated, at least in the neighborhood of the aperture, by the static solution, and on the fact that the far field may be obtained from the aperture field. The static solution is the solution of the above-mentioned potential problem. The first method is the so-called static method of Rayleigh [1] and the second is the more powerful variational procedure of Schwinger and Levine [4a]. In the first method it is necessary to know only a gross characteristic of the static aperture field, namely

the conductivity of the opening. The second method, on the other hand, requires a detailed knowledge of the low-frequency aperture field, for which the static field in the aperture is useful as a first approximation. Since explcit expressions for the conductivity of the opening and for the aperture field are obtained from our solution of the steady-flow potential problem, both procedures may be successfully employed. In this report, however, primarily for the sake of brevity, we obtain the approximate far fields by means of Rayleigh's procedure.

The solutions of many potential problems in two dimensions are available in virture of the applicability of conformal mapping methods of function theory. In three dimensions, however, the number of known exact solutions is small. In particular, the solutions of the most general aperture problems found in the literature were already employed by Rayleigh [1] when he introduced his static method.

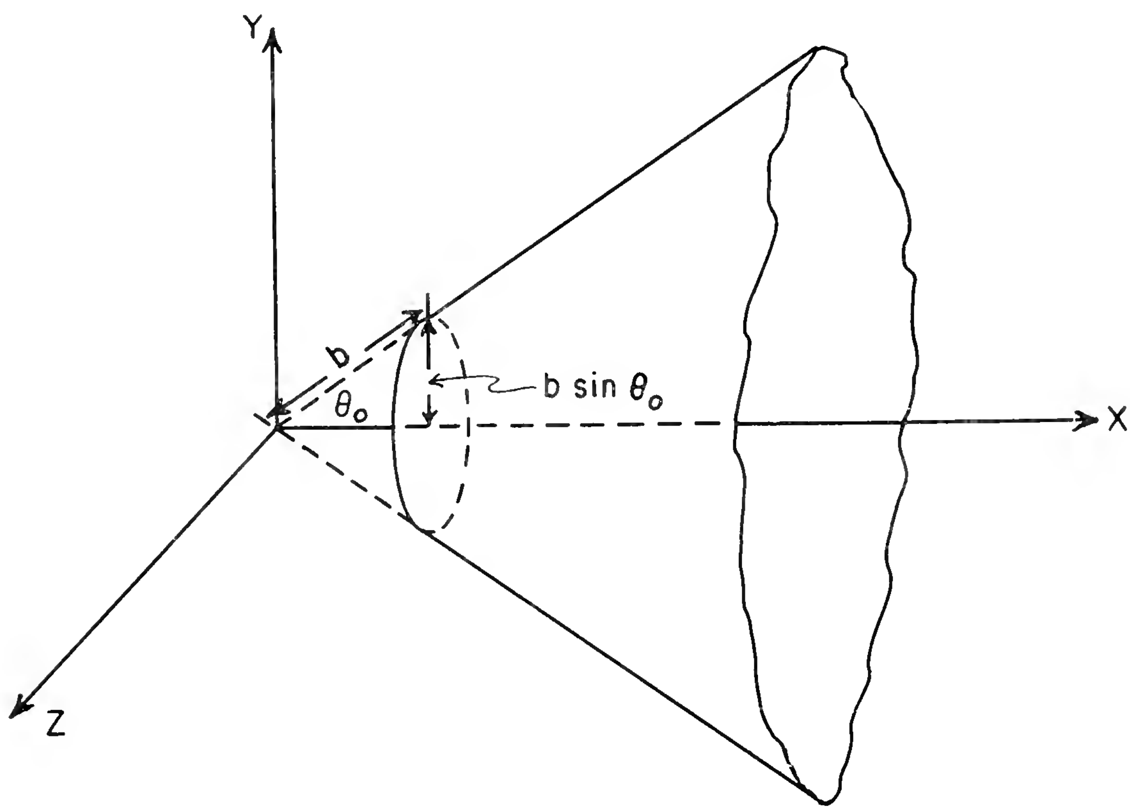

Figure 1

These are solutions of problems involving potential flow through circular or more generally elliptical apertures in rigid plane screens. In an electrostatic context, the potential arising from the natural charge distribution of a finite conical cup has been obtained previously by one of the present authors[6]; similar methods are employed in the present work. It should be mentioned, in addition, that Schwinger and Levine have given an exact solution for the problem of radiation of sound from an unflanged circular pipe[4a].

We summarize briefly the ensuing argument. Section 2 will be devoted to a detailed statement of the problem. In Section 3 we discuss some properties of the Legendre function and certain related functions by way of preparation for the considerations of Section 4 where we give the details of the construction and discuss the analytical properties of the various integral representations of the solution. Section 5 is devoted to giving the eigenfunction representations of the solution and also to obtaining an expression for the conductivity of the opening. In Section 6 we give explicit formulas for the behavior of the velocity of the fluid near the circular edge and in Section 7 we make use of this behavior in proving the uniqueness of the solution. Section 8 is devoted to an application of the Rayleigh static method; we obtain the approximate far field 'outside' the cone that results from a given exciting field 'inside' the cone. Appendices I - IV are added to dispose of certain details arising from the considerations of Sections 3, 4, 5 and 6.

2. Detailed Statement of the Problem

To fix the geometry of the problem consider a conical surface with apex at the origin and with its axis of symmetry lying on the positive X-axis.

Let θ_o be the angle from the axis of the cone to a generator [see Fig. 1]. The conical pipe is that part of this surface defined by the relations $r > b$, $\theta = \theta_o$ where r is this distance from the origin and θ is measured from the positive X-axis. When $\theta_o = \pi/2$ the conical pipe reduces to a plane with a circular aperture. If we assume that the spherical coordinates (r, θ, ϕ) are related to the cartesian coordinates (x,y,z) by the equations

$$(2.1) \qquad \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \cos \phi \\ z &= r \sin \theta \sin \phi, \qquad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \end{aligned}$$

then the conical pipe, C, is determined by the following relations:

$$(2.2) \qquad \begin{aligned} & r > b, \\ & x = r \cos \theta_o \\ & y = r \sin \theta_o \cos \phi \\ & z' = r \sin \theta_o \sin \phi. \end{aligned}$$

The angle θ_o is assumed to be greater than 0 and less than π.

Our problem is to determine the velocity potential for the axially symmetric, steady-state, irrotational flow of an incompressible, non-viscous fluid through the aperture of the cone. More explicitly we seek a solution U of the ϕ-independent Laplace's equation

$$(2.3) \qquad \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial U(r,\theta)}{\partial r} \right] + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial U(r,\theta)}{\partial \theta} \right] = 0$$

which satisfies the following conditions:

(2.4)

(a) $U(r,\theta)$ is continuous for all r and θ, $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ except across the conical pipe, C.

(b) $\dfrac{\partial U(r,\theta)}{\partial \theta}$ and $\dfrac{\partial^2 U(r,\theta)}{\partial \theta^2}$ are continuous for all r and θ, $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ except at the circular edge. $\dfrac{\partial U(r,\theta)}{\partial r}$ is continuous for all r and θ except near the circular edge and across the cone.

(c) $\frac{\partial U}{\partial \theta} = 0$ on C.

(2.4)

(d) $U^{(1)} = U_o^{(1)} + \frac{f}{4\pi \sin^2(\theta_o/2)} \cdot \frac{1}{r}$ as $r \to \infty$ for θ in the range $0 \leq \theta \leq \theta_o$, where $U_o^{(1)}$ and f are constant.

If the density of the fluid is assumed to be unity it is easy to verify that f is the total incoming flux at infinity in the region $0 \leq \theta \leq \theta_o$. This assertion may be demonstrated by evaluating the following integral:

$$\int_{S_{\theta_o}} U_r d\sigma = \int_0^{\theta} \int_0^{2\pi} \frac{\partial U}{\partial r} r^2 \sin\theta \, d\theta \, d\phi.^{*} \tag{2.5}$$

Here S_{θ_o} is a spherical cap determined by the cone C and a large sphere S centered at the origin, U_r is the radial component of the velocity, and $d\sigma = r^2 \sin\theta \, d\theta \, d\phi$ is the differential area expressed in spherical polar coordinates.

We prefer to leave open the specification of the behavior of U and ∇U near the circular edge of the cone. It suffices to say here that the solution we obtain will be such that, in any axial cross-section, U is bounded and $|\nabla U| = O(\delta^{-1/2})$, where δ is the distance from the edge.

3. Some Preliminary Results

This section consists of a summary and discussion of results of varied character. The unifying feature of these results is their usefulness in Section 4, where we give the details of the construction of the integral representations of our solution, and also in subsequent sections where we investigate the nature of the solution. As we shall see, our integral representations will involve the Legendre functions $P_{(2\nu-1)/2}(\pm \cos\theta)$ and their θ-derivatives regarded as functions of the real variable θ, $0 \leq \theta \leq \pi$, and the complex subscript ν. In the present section, after discussing some

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* Throughout this paperit is assumed that the velocity of the fluid is given by ∇U. It should be noted that some authors [cf. Lamb [5]] employ $\nabla(-U)$ for this quantity.

analytical properties of these functions and summarizing pertinent formulas, we obtain infinite product representations at $dP_{(2\nu-1)/2}(\pm \cos\theta)/d\theta$, for fixed θ and variable ν. These infinite product representations are then employed in the construction of the meromorphic functions $K^{+}(\nu,\theta_0)$ and $K^{-}(\nu,\theta_0)$ [see Eqs. (3.21) and (3.22)] which also appear in our integral representations. Some analytical properties of $K^{\pm}(\nu,\theta_0)$, regarded as functions of ν, are also noted.

The basic functions in our integral representations are product solutions of Eq. (2.3) which are obtained by the usual separation of variables procedure. If we write $U(r,\theta) = R(r)T(\theta)$ we have, after substituting for $U(r,\theta)$ in Eq. (2.3) and setting the separation constant equal to $(4\nu^2-1)/4$,

$$\frac{d}{dr}\left[r^2 \frac{dR}{dr}\right] - \frac{4\nu^2-1}{4} R = 0 \tag{3.1}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dT}{d\theta}\right] + \frac{4\nu^2-1}{4} T = 0, \tag{3.2}$$

where ν is, in general, complex. The integral representations will be formed in terms of the following products of the solutions of Eqs. (3.1) and (3.2):

$$\begin{aligned} w(r,\theta) &= r^{-(2\nu+1)/2} P_{(2\nu-1)/2}(\cos\theta), && 0 \le \theta \le \theta_0 \\ v(r,\theta) &= r^{-(2\nu+1)/2} P_{(2\nu-1)/2}(-\cos\theta), && \theta_0 \le \theta \le \pi. \end{aligned} \tag{3.3}$$

The Legendre function of the first kind, $P_{(2\nu-1)/2}(x)$, is the only solution of Eq. (3.2) which is regular at $x = 1$ for arbitrary complex values of ν, and is employed with a view to satisfying the regularity conditions listed in (2.4a) and (2.4b). $P_{(2\nu-1)/2}(\cos\theta)$ and $P_{(2\nu-1)/2}(-\cos\theta)$, appearing in Eq. (3.3) are integral functions of ν for θ in $0 \le \theta < \pi$ and θ in $0 < \theta \le \pi$ respectively. The Wronskian

$$W(\nu,\theta) = P_{(2\nu-1)/2}(-\cos\theta)\,\frac{dP_{(2\nu-1)/2}(\cos\theta)}{d\theta} - P_{(2\nu-1)/2}(\cos\theta)\,\frac{dP_{(2\nu-1)/2}(-\cos\theta)}{d\theta}$$

of $P_{(2\nu-1)/2}(-\cos\theta)$ and $P_{(2\nu-1)/2}(\cos\theta)$ is easily seen to be [cf. [2]p. 63]

$$W(\nu,\theta) = \frac{2\cos\pi\nu}{\pi\sin\theta}\,. \tag{3.4}$$

We turn now to the problem of obtaining an infinite product representation of the θ-derivative of the Legendre function regarded as a function of ν for fixed θ ($= \theta_o$). To this end we investigate the zeros of $P'_{(2\nu-1)/2}(\cos\theta_o)$ and $P'_{(2\nu-1)/2}(-\cos\theta_o)$, where

$$P'_{(2\nu-1)/2}(\pm\cos\theta_o) = \lim_{\theta \to \theta_o}\left[\frac{dP_{(2\nu-1)/2}(\pm\cos\theta)}{d\theta}\right], \quad 0 < \theta_o < \pi \tag{3.5}$$

(the prime will henceforth be reserved for differentiation with respect to θ).

Now from [2], p. 63, we have

$$P'_{(2\nu-1)/2}(\cos\theta_o) = -\left(\frac{2\nu-1}{2}\right)\left(\frac{2\nu+1}{2}\right)P^{-1}_{(2\nu-1)/2}(\cos\theta_o), \tag{3.6}$$

where $P^{\mu}_{\nu}(\cos\theta_o)$ is the associated Legendre function of the ν-th degree and μ-th order. Furthermore, from [2], p. 70, Section h, it is known that $P^{-1}_{(2\nu-1)/2}(\pm\cos\theta_o)$ is an integral function of ν and, in addition, has infinitely many zeros all of which are real and simple. Thus, employing the relation

$$P^{\mu}_{(-2\nu-1)/2}(\cos\theta) = P^{\mu}_{(2\nu-1)/2}(\cos\theta) \tag{3.7}$$

[cf. [2], p. 62] and taking θ as fixed, $0 \le \theta < \pi$, we conclude from Eq. (3.6): a) that $P'_{(2\nu-1)/2}(\cos\theta)$ is an integral function of ν; b) that $P'_{(2\nu-1)/2}(\cos\theta)$ is an even function of ν, i.e., that

$$P'_{(2\nu-1)/2}(\cos\theta) = P'_{(-2\nu-1)/2}(\cos\theta); \tag{3.8}$$

and c) that $P'_{(2\nu-1)/2}(\cos\theta) = 0$ has infinitely many roots all of which are real and simple. In particular, when $\theta = \frac{\pi}{2}$

$$(3.9) \qquad P^{-1}_{(2\nu-1)/2}(0) = \frac{\pi^{1/2}}{2\Gamma(\frac{5}{4}+\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{\nu}{2})} ,$$

[cf. [2] p. 63]. From Eq. (3.6) it then follows that

$$(3.10) \qquad P'_{(2\nu-1)/2}(0) = \frac{\pi^{1/2}}{2} \frac{(2\nu+1)/2 \cdot (2\nu-1)/2}{\Gamma(\frac{5}{4}+\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{\nu}{2})} = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{1}{4}-\frac{\nu}{2})} .$$

The p-th positive zero, $\nu_p(C)$, of $P'_{(2\nu-1)/2}(0)$ is clearly

$$(3.11) \qquad \nu_p = 2(p - \tfrac{3}{4}), \qquad p = 1, 2, \cdots .$$

For more general angles θ_o, the following is known about the asymptotic behavior of the zeros of $P'_{(2\nu-1)/2}(\cos\theta_o)$. Let ε be any arbitrarily small positive number. Then for θ_o in the range $0 < \varepsilon \leq \theta_o \leq \pi-\varepsilon$ it can be shown [[3], pp.404-406] that the p-th positive zero of $P'_{(2\nu-1)/2}(\cos\theta_o)$ is of the form

$$(3.12) \qquad \nu_p(\theta_o) \sim \frac{\pi}{\theta_o} \left[(n_p(\theta_o) - \frac{3}{4}\right] + \frac{C(\theta_o)}{p} , \qquad p \to \infty,$$

where $n_p(\theta_o)$ is an integer and $C(\theta_o)$ is bounded and independent of p. Since it can be assumed that $\nu_p(\theta_o)$ is continuous in θ_o, $0 < \varepsilon_1 \leq \theta_o \leq \pi-\varepsilon$ and since $\nu_p(\pi/2) = 2(p - \frac{3}{4})$ [cf. Eq. (3.11)] it follows that the p-th positive zero is given by

$$(3.12a) \qquad \nu_p(\theta_o) \sim \frac{\pi}{\theta_o} (p - \tfrac{3}{4}) + \frac{C(\theta_o)}{p} \qquad p \to \infty.$$

Now, in virture of the fact that $P'_{(2\nu-1)/2}(\cos\theta_o)$ is an even function of ν [see Eq. (3.7)] and that the exponential order of $P'_{(2\nu-1)/2}(\cos\theta_o)$ is unity*

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* See footnote in Appendix II page 58.

we may invoke the Hadamard factorization theorem ([5] pp. 250-251) and write

$$P'_{(2\nu-1)/2}(\cos\theta_0) = P'_{-\frac{1}{2}}(\cos\theta_0)\, \Pi(\nu,\theta_0)\, \Pi(-\nu,\theta_0), \tag{3.13}$$

where

$$\Pi(\nu,\theta_0) = \prod_{p=1}^{\infty} \left\{ \left(1 + \frac{\nu}{\nu_p(\theta_0)}\right) \exp\left(- \frac{\nu}{\nu_p(\theta_0)}\right) \right\}. \tag{3.14}$$

Note that for any θ_0, $0 < \theta_0 < \pi$ we have

$$\nu_1(\theta_0) = - \frac{1}{2}, \tag{3.15}$$

[cf. Eq. (3.6)]. Similarly we may write for $P'_{(2\nu-1)/2}(-\cos\theta_0)$

$$P'_{(2\nu-1)/2}(\cos\chi_0) = P'_{-\frac{1}{2}}(\cos\chi_0)\, \Pi(\nu, \chi_0)\, \Pi(-\nu, \chi_0) \tag{3.16}$$

where we have set

$$\chi_0 = \pi - \theta_0 . \tag{3.17}$$

As we have mentioned in Section 1, our solution will be constructed by a variant of the well-known Wiener-Hopf procedure. A fundamental step in this procedure is factorization of a given analytic function into the product of two functions each of which is regular, zeroless, and of algebraic growth in appropriate half-planes which overlap in a suitable manner. In the present case this function turns out to be

$$K(\nu,\theta_0) = \frac{P'_{(2\nu-1)/2}(\cos\theta_0)P'_{(2\nu-1)/2}(-\cos\theta_0)}{W(\nu,\theta_0)}, \tag{3.18}$$

which, when we use Eq. (3.4), becomes

$$K(\nu,\theta_0) = \frac{\sin\theta_0}{2}\, \Gamma(\tfrac{1}{2} + \nu)\Gamma(\tfrac{1}{2} - \nu)P'_{(2\nu-1)/2}(-\cos\theta_0)P'_{(2\nu-1)/2}(\cos\theta_0), \tag{3.19}$$

in virtue of the identity $\pi/\cos\pi\nu = \Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu)$. By way of preparation for the considerations of Section 4 we shall prove here that $K(\nu,\theta_o)$ can be written as

$$K(\nu,\theta_o) = K^+(\nu,\theta_o)K^-(\nu,\theta_o), \tag{3.20}$$

where

(3.21)
(a) $K^+(\nu,\theta_o)$ is regular and zeroless in the half-plane $\operatorname{Re}\nu > -\frac{1}{2}$;

(a') $K^-(\nu,\theta_o)$ is regular and zeroless in the half-plane $\operatorname{Re}\nu < \frac{1}{2}$;

(b) $K^+(\nu,\theta_o) \sim k^+(\theta_o, \chi_o)\nu^{1/2}$ as $|\nu + \frac{1}{2}| \to \infty$ in the angular region $|\arg(\nu+\frac{1}{2})| < \pi$;

(b') $K^-(\nu,\theta_o) \sim k^-(\theta_o, \chi_o)(-\nu)^{1/2}$ as $|\nu - \frac{1}{2}| \to \infty$ in the angular region $0 < \arg(-\nu+\frac{1}{2}) < 2\pi$ where $k^-(\theta_o, \chi_o) = k^+(\theta_o, \chi_o)$ and $\chi_o = \pi - \theta_o$;

and where, in addition,

(3.21) (c) $K^-(\nu,\theta_o) = K^+(-\nu,\theta_o)$.

The quantity $k^+(\theta_o, \chi_o)$ is independent of ν; an explicit expression for it will be given below. We turn now to the construction of $K^+(\nu,\theta_o)$ and $K^-(\nu,\theta_o)$. Let us define $K^+(\nu,\theta_o)$ as follows:

$$K^+(\nu,\theta_o) = \left[\frac{\sin\theta_o}{2} P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)\right]^{\frac{1}{2}} \Gamma(\tfrac{1}{2}+\nu)\Pi(\nu,\theta_o)\Pi(\nu,\chi_o)e^{t\nu} \tag{3.22}$$

where t is a constant to be determined later. It is clear that $K^+(\nu,\theta_o)$ satisfies condition (3.21a). Furthermore if $K^-(\nu,\theta_o)$ is defined as $K^+(-\nu,\theta_o)$ then it is easily verified that Eq. (3.20) is satisfied. If now the parameter t can be chosen in such a manner that condition (3.21b) is satisfied for $K^+(\nu,\theta_o)$ then condition (3.21b') is automatically satisfied by $K^-(\nu,\theta_o)$.

We consider now the determination of the constant t. To this end it is necessary to know the asymptotic behavior of $\Pi(\nu,\theta_o)$ and $\Pi(\nu, \chi_o)$ in the angular region $|\arg(\nu + \frac{1}{2})| \leq \pi$. It can be shown (Appendix I) that

$$(3.23) \quad \Pi(\nu,\theta_o) \sim \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{\theta_o}{\pi}\nu + \frac{1}{4})} L(\theta_o) \exp\left\{\nu\left[\frac{\theta_o}{\pi} M(\theta_o) + \Psi(\tfrac{1}{4})\right]\right\} \quad |\arg(\nu + \tfrac{1}{2})| < \pi,$$

where

$$\Psi(z) = \Gamma'(z)/\Gamma(z),$$

$$(3.24) \qquad L(\theta_o) = \prod_{p=1}^{\infty}\left\{\frac{(p - \frac{3}{4})}{\frac{\theta_o}{\pi}\nu_p(\theta_o)}\right\},$$

$$M(\theta_o) = \sum_{p=1}^{\infty}\left[\frac{1}{p - \frac{3}{4}} - \frac{1}{(\theta_o/\pi)\nu_p(\theta_o)}\right].$$

Note that when $\theta_o = \frac{\pi}{2}$ we have, by virtue of Eq. (3.11),

$$(3.25) \qquad L(\tfrac{\pi}{2}) = 1, \qquad M(\tfrac{\pi}{2}) = 0.$$

It can be shown (see Appendix I) that for this case (3.23) is an equality.

From the definition of $\Pi(\nu, \chi_o)$ and Eq. (3.23) we have

$$(3.26) \qquad \Pi(\nu, \chi_o) \sim \frac{\Gamma(1/4)}{\Gamma(\frac{\chi_o}{\pi}\nu + \frac{1}{4})} L(\chi_o)\exp\left\{\nu\left[\frac{\chi_o}{\pi} M(\chi_o) + \Psi(1/4)\right]\right\},$$

for $|\arg(\nu + \frac{1}{2})| < \pi$. The asymptotic behavior of $K^+(\nu,\theta_o)$ in this region is therefore

$$K^+(\nu,\theta_o) \sim \left[\frac{\sin\theta_o}{2} P^1_{-\frac{1}{2}}(\cos\theta_o)P^1_{-\frac{1}{2}}(\cos\chi_o)\right]^{\frac{1}{2}}\left\{\frac{\Gamma(\frac{1}{2} + \nu)\Gamma^2(\frac{1}{4})}{\Gamma(\frac{\theta_o}{\pi}\nu + \frac{1}{4})\Gamma(\frac{\chi_o}{\pi}\nu + \frac{1}{4})}\right\} L(\theta_o)L(\chi_o)$$

$$(3.27) \qquad \cdot \exp\left\{\nu\left[t + \left(\frac{\theta_o}{\pi} M(\theta_o) + \frac{\chi_o}{\pi} M(\chi_o) + \frac{\chi_o + \theta_o}{\pi}\Psi(1/4)\right)\right]\right\}, \quad \left|\arg(\nu + \tfrac{1}{2})\right| \leq \pi .$$

Using Stirling's formula

$$\Gamma(\nu) \sim \sqrt{2\pi}\ \exp\left[(\nu - \tfrac{1}{2})\log \nu - \nu\right], \qquad |\arg \nu| < \pi,$$

we have

$$\frac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(\frac{\theta_o}{\pi}\nu + \frac{1}{4})\Gamma(\frac{\chi_o}{\pi}\nu + \frac{1}{4})} \sim \frac{1}{\pi\sqrt{2}}(\theta_o\,\chi_o)^{\frac{1}{4}}\,\nu^{\frac{1}{2}}\exp\left\{-\nu\left(\frac{\theta_o}{\pi}\log\frac{\theta_o}{\pi} + \frac{\chi_o}{\pi}\log\frac{\chi_o}{\pi}\right)\right\},$$

and hence we obtain in the region $|\arg(\nu + \frac{1}{2})| < \pi$

(3.28) $$K^+(\nu,\theta_o) \sim \kappa^+(\theta_o\,\chi_o)\nu^{\,1/2}$$

(3.29) $$\kappa^+(\theta_o,\chi_o) = \frac{1}{2\pi}\left[\sin\theta_o P^{\,'}_{-\frac{1}{2}}(\cos\theta_o)P^{\,'}_{-\frac{1}{2}}(\cos\chi_o)(\theta_o\,\chi_o)^{\frac{1}{2}}\right]^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4})L(\theta_o)L(\chi_o),$$

where we have chosen $t = t(\theta_o)$ to be

(3.30) $$t(\theta_o) = -\left[\frac{\theta_o}{\pi}M(\theta_o) + \frac{\chi_o}{\pi}M(\chi_o) + \left(\frac{\theta_o + \chi_o}{\pi}\right)\Psi(\tfrac{1}{4}) - \frac{\theta_o}{\pi}\log\frac{\theta_o}{\pi} - \frac{\chi_o}{\pi}\log\frac{\chi_o}{\pi}\right].$$

We have thus determined t and $\kappa^+(\theta_o, \chi_o)$.

When $\theta_o = \chi_o = \pi/2$ $K^{\pm}(\nu,\theta_o)$ can be expressed as a quotient of gamma functions. The proof of this assertion follows. Note that when $\theta_o = \chi_o = \pi/2$ we have by virtue of Eq. (3.25)

(3.31) $$t(\pi/2) = -\left(\Psi(1/4) + \log 2\right).$$

Since, as already mentioned, (3.23) is an equality for $\theta_o = \chi_o = \pi/2$ we conclude that

(3.32) $$\Pi(\nu, \tfrac{\pi}{2}) = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4} + \frac{\nu}{2})}\ \exp\left[\tfrac{1}{2}\,\Psi(\tfrac{1}{4})\right].$$

It follows then from Eq. (3.22) that*

$$(3.33)\qquad K^{+}(\nu,\tfrac{\pi}{2}) = i\,\frac{P'_{-\frac{1}{2}}(0)\,\Gamma^{2}(\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}+\nu)\exp\left[\nu\left(t(\tfrac{\pi}{2}) + \Psi(\tfrac{1}{4})\right)\right]}{2^{1/2}\,\Gamma^{2}(\tfrac{1}{4}+\tfrac{\nu}{2})} .$$

This expression for $K^{+}(\nu, \frac{\pi}{2})$ can be further simplified. Since

$$(3.34)\qquad P'_{-\frac{1}{2}}(0) = \frac{2\pi^{1/2}}{\Gamma^{2}(\frac{1}{4})},$$

[cf. Eq. (3.10)] and since

$$(3.35)\qquad \Gamma(2z) = (2\pi)^{-1/2}2^{(4z-1)/2}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}),$$

[cf. [2] p. 1], implies that

$$(3.36)\qquad \Gamma(\tfrac{1}{2}+\nu) = (2\pi)^{-1/2}\,2^{\nu}\,\Gamma(\tfrac{1}{4}+\tfrac{\nu}{2})\,\Gamma(\tfrac{3}{4}+\tfrac{\nu}{2}),$$

we have, on writing $2^{\nu} = \exp(\nu\log 2)$ and employing Eq. (3.33),

$$(3.37)\qquad K^{+}(\nu, \tfrac{\pi}{2}) = i\,\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2})} .$$

Since $K^{-}(\nu, \frac{\pi}{2}) = K^{+}(-\nu, \frac{\pi}{2})$ we have

$$(3.38)\qquad K^{-}(\nu, \tfrac{\pi}{2}) = i\,\frac{\Gamma(\frac{3}{4}-\frac{\nu}{2})}{\Gamma(\frac{1}{4}-\frac{\nu}{2})} .$$

---

* Note that $P'_{-\frac{1}{2}}(-\cos\theta_0) = \left.\frac{dP_{-1/2}(-\cos\theta)}{d\theta}\right|_{\theta=\theta_0} = -\left.\frac{d}{d\chi}P_{-1/2}(\cos\chi)\right|_{\chi=\chi_0}$, where $\chi_0 = \pi - \theta_0$. On setting $\theta_0 = \chi_0 = \pi/2$, the first factor in the right hand side of Eq. (3.22) becomes $2^{-1}\,i\,P'_{-\frac{1}{2}}(0)$ where $i = \sqrt{-1}$.

As a check on our procedure we shall obtain $K^+(\nu, \frac{\pi}{2})$ and $K^-(\nu, \frac{\pi}{2})$ more directly. After setting $\theta_o = \frac{\pi}{2}$ in Eq. (3.22) and substituting for $P'_{(2\nu-1)/2}(0)$ from Eq. (3.10) we get

$$(3.39) \qquad K(\nu,\theta_o) = -2\pi \frac{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}{\Gamma^2(\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{1}{4}-\frac{\nu}{2})^2} \, .$$

Employing Eq. (3.36) we then have

$$(3.40) \qquad K(\nu,\theta_o) = \left[i \frac{\Gamma(\frac{3}{4}+\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2})} \right] \left[i \frac{\Gamma(\frac{3}{4}-\frac{\nu}{2})}{\Gamma(\frac{1}{4}-\frac{\nu}{2})} \right] \, .$$

The first factor on the right-hand side of Eq. (3.40) is clearly $K^+(\nu, \frac{\pi}{2})$, and the second $K^-(\nu, \frac{\pi}{2})$ [cf. Eqs. (3.37), (3.38)].

Note that $K^+(\nu,\theta_o)$ and $K^-(\nu,\theta_o)$ are purely imaginary when $\theta_o = \pi/2$ and ν is real. It is useful to observe here for the purposes of Sections 5, 6 and 7 that this assertion is true for all θ_o in the range $0 < \theta_o < \pi$. To prove this statement we note from the form of $K^+(\nu,\theta_o)$ in Eq. (3.22) that it is only necessary to prove that $\left[P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)\right]^{1/2}$ is purely imaginary. Now from [2] p. 74 we have

$$(3.41) \qquad P_{-\frac{1}{2}}(\cos\theta_o) = 1 + \frac{1^2}{2^2}\sin^2\left(\frac{\theta_o}{2}\right) + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\left(\frac{\theta_o}{2}\right) + \cdots .$$

It is now easily verified that $P'_{-\frac{1}{2}}(\cos\theta_o) > 0$, $0 \le \theta_o < \pi$. Finally we observe that [cf. Eq. (3.5)]

$$(3.42) \qquad P'_{-\frac{1}{2}}(\cos\chi_o) = \frac{d}{d\theta_o} P_{-\frac{1}{2}}(\cos\chi_o) = -\frac{d}{d\chi_o} P_{-\frac{1}{2}}(\cos\chi_o),$$

since $\chi_o = \pi - \theta_o$. It follows immediately that

$$\left[P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)\right]^{\frac{1}{2}} = \pm\, i \left[\frac{dP_{-1/2}(\cos\theta_o)}{d\theta_o}\;\frac{dP_{-1/2}(\cos\chi_o)}{d\chi_o}\right]^{\frac{1}{2}}$$

where the term in braces on the right-hand side is positive. It is a matter of indifference which sign we employ in the right-hand side of the last equation. In the remainder of the report we shall always assume that

$$\left[P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)\right]^{\frac{1}{2}} = +\, i \left[\frac{dP_{-1/2}(\cos\theta_o)}{d\theta_o}\;\frac{dP_{-1/2}(\cos\chi_o)}{d\chi_o}\right]^{\frac{1}{2}} . \tag{3.43}$$

Thus $K^+(\nu,\theta_o)$ and $K^-(\nu,\theta_o)$ are imaginary quantities when $0 < \theta_o < \pi$ and ν is real. Indeed we can further specify that $K^+(\nu,\theta_o)$ is positive imaginary when ν is positive, and $K^-(\nu,\theta_o)$ is positive imaginary when ν is negative [cf. Eqs. (3.22), (3.14) and (3.21c)].

4. Integral Representations of the Solution

In this section we obtain integral representations of the solution of the problem. We construct these integral representations on the basis of a formal procedure related to the well-known Wiener-Hopf procedure. We then verify that the representations of the solution and its various derivatives are convergent and that they indeed provide a solution of the problem. In Section 7 conditions are given which insure the uniqueness of the solution up to an arbitrary constant.

For convenience we shall assume that the distance measured along a generator from the origin to the opening of the cone is unity. The integral representations for the case where this distance is b may be obtained from the preceding case by replacing r by r/b in the integrand.

Let $U(r,\theta)$ be a solution of the problem and write

$$U^1(r,\theta) \equiv U(r,\theta), \qquad r \geq 0, \quad 0 \leq \theta \leq \theta_o$$

(4.1)

$$U^2(r,\theta) \equiv U(r,\theta), \qquad r \geq 0, \quad \theta_o \leq \theta \leq \pi.$$

The function $U^1(r,\theta)$ thus represents the solution inside the cone (i.e., where $0 \leq \theta \leq \theta_o$) whereas $U^2(r,\theta)$ represents the solution outside the cone (i.e., where $\theta_o \leq \theta \leq \pi$). We begin by assuming that $U^1(r,\theta)$ and $U^2(r,\theta)$ have integral representations of the following form:

$$(4.2) \qquad U^1(r,\theta) = \frac{N}{2\pi i} \int_{C_\rho'} r^{(-2\nu-1)/2} P_{(2\nu-1)/2}(\cos\theta) A(\nu) d\nu, \quad r \geq 0,\ 0 \leq \theta \leq \theta_o,$$

$$(4.2') \qquad U^2(r,\theta) = \frac{N}{2\pi i} \int_{C_\rho'} r^{(-2\nu-1)/2} P_{(2\nu-1)/2}(-\cos\theta) B(\nu) d\nu, \quad r \geq 0,\ \theta_o \leq \theta \leq \pi.$$

Note that the expressions for $U^1(r,\theta)$ and $U^2(r,\theta)$ involve the functions $w(r,\theta) = r^{(-2\nu-1)/2} P_{(2\nu-1)/2}(\cos\theta)$ and $v(r,\theta) = r^{(-2\nu-1)/2} P_{(2\nu-1)/2}(-\cos\theta)$ respectively. It will be recalled that these functions, which were discussed at the beginning of Section 3, are the product solutions of the axially symmetric potential equation (2.3) obtained by the method of separation of variables. The former is regular in θ when θ is in the interval $0 \leq \theta \leq \theta_o$, and the latter is regular in θ for θ in the interval $\theta_o \leq \theta \leq \pi$. The functions $A(\nu)$ and $B(\nu)$ are analytic functions of ν to be determined from the conditions of the problem. The quantity N is employed for purposes of normalization. It will be determined [see Eq. (5.14)] by the requirement that the flux entering the conical region $0 \leq \theta \leq \theta_o$ be f [cf. Eqs. (2.4d) and (2.5)]. The contour C_ρ' [cf. Fig. 2] is the curve traced from the point $-(\frac{1}{2} + i\infty)$ to $-\frac{1}{2} + i\infty$ along the line $\mathrm{Re}\ \nu = -1/2$ except near the point $\nu = -1/2$ which is circumvented by a small semicircle traced in a clockwise fashion. The choice of C_ρ' is dictated by the requirement that $U^1(r,\theta)$ approach a constant at infinity [cf. Eq. (2.4d)]. That the present choice of C_ρ' suffices to satisfy

this requirement will become apparent later.

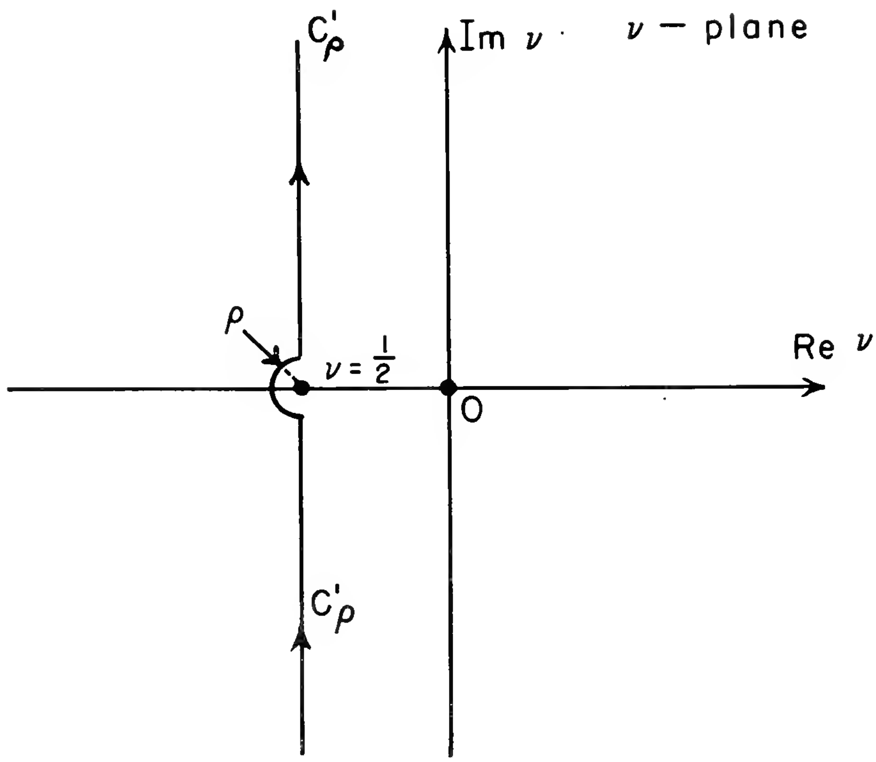

Figure 2

Our object now is to determine the functions $A(\nu)$ and $B(\nu)$ from the conditions of the problem. Let us assume, subject to later verification, that the integrals of Eqs. (4.2) and (4.2') are convergent and that it is permissible to take the necessary limits and derivatives under the integral signs. It follows immediately that $U^1(r,\theta)$ and $U^2(r,\theta)$ satisfy the potential equation in the regions $0 \leq \theta < \theta_o$ and $\theta_o < \theta \leq \pi$ respectively. Furthermore, if $U(r,\theta)$ is to be a solution of the problem it is necessary that

$$U^1(r,\theta)\Big|_{\theta\uparrow\theta_o} = U^2(r,\theta)\Big|_{\theta\downarrow\theta_o}, \qquad 0 \leq r < 1,$$

(4.3)

$$\frac{\partial U^1(r,\theta)}{d\theta}\Big|_{\theta\uparrow\theta_o} = \frac{\partial U^2(r,\theta)}{d\theta}\Big|_{\theta\downarrow\theta_o} \qquad r \geq 0.$$

In order to satisfy these conditions let us first write*

$$A(\nu) = \frac{P'_{(2\nu-1)/2}(-\cos\theta_o)}{W(\nu,\theta_o)} \; D(\nu)$$

(4.4)

$$B(\nu) = \frac{P'_{(2\nu-1)/2}(\cos\theta_o)}{W(\nu,\theta_o)} \; D(\nu),$$

where

(4.5) $$W(\nu,\theta_o) = P_{(2\nu-1)/2}(-\cos\theta_o)P'_{(2\nu-1)/2}(\cos\theta_o) - P_{(2\nu-1)/2}(\cos\theta_o)P'_{(2\nu-1)/2}(-\cos\theta_o)$$

$$= \frac{2\cos\pi\nu}{\pi\sin\theta_o}$$

is the Wronskian of $P_{(2\nu-1)/2}(-\cos\theta_o)$ and $P_{(2\nu-1)/2}(\cos\theta)$ evaluated at $\theta = \theta_o$ [cf. Eq. (3.4)]. Under these circumstances Eqs. (4.2) and (4.2') become

(4.6) $$U^1(r,\theta) = \frac{N}{2\pi i}\int_{C'_\rho} r^{(-2\nu-1)/2}\, \frac{P'_{(2\nu-1)/2}(-\cos\theta_o)P_{(2\nu-1)/2}(\cos\theta)}{W(\nu,\theta_o)} \; D(\nu)d\nu,$$

$$r \geq 0,\; 0 \leq \theta \leq \theta_o$$

(4.6') $$U^2(r,\theta) = \frac{N}{2\pi i}\int_{C'_\rho} r^{(-2\nu-1)/2}\, \frac{P'_{(2\nu-1)/2}(\cos\theta_o)P_{(2\nu-1)/2}(-\cos\theta)}{W(\nu,\theta_o)} \; D(\nu)d\nu,$$

$$r \geq 0,\; \theta_o \leq \theta \leq \pi.$$

It is now easy to verify that the second of the conditions (4.3) is satisfied, since if we differentiate with respect to θ and set $\theta = \theta_o$, the integrands in (4.6) and (4.6') become identical. In order that the first of

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*It will be recalled that $P'_{(2\nu-1)/2}(\cos\theta_o)$ and $P'_{(2\nu-1)/2}(-\cos\theta_o)$ are used to denote $[dP_{(2\nu-1)/2}(\cos\theta)]/d\theta$ and $[dP_{(2\nu-1)/2}(-\cos\theta_o)]/d\theta$ evaluated at $\theta = \theta_o$.

the conditions (4.3) be satisfied it is necessary that

$$(4.7) \qquad 0 = \int_{C_\rho'} r^{(-2\nu-1)/2}\, D(\nu)d\nu, \qquad 0 \le r < 1.$$

This relation is obtained by subtracting Eq. (4.6') from Eq. (4.6), evaluating the result at $\theta = \theta_o$, and employing Eq. (4.5).

If in the region to the left of the contour C_ρ' we assume that $D(\nu) \sim c|\nu + \frac{1}{2}|^{-p}$, $p > 0$, as $|\nu + \frac{1}{2}| \to \infty$, c being a constant independent of arg ν, then the integral

$$\int_{C_\rho'} r^{(-2\nu-1)/2} D(\nu)d\nu = \int_{C_\rho'} \exp\left\{-[(\nu + \tfrac{1}{2})\log r]\right\} D(\nu)d\nu, \quad 0 \le r < 1$$

may be evaluated by residues in the left half-plane. If in addition we assume that $D(\nu)$ is regular in this half-plane then the result after evaluating by residues is zero. To indicate the regularity of $D(\nu)$ to the left of the contour C_ρ' we shall henceforth write

$$(4.8) \qquad D^-(\nu) = D(\nu).$$

In order to satisfy the boundary conditions on the cone [cf. Eq. (2.4)] it is clear from Eqs. (4.6), (4.6') and (4.8) that we must have for $r > 1$

$$(4.9) \quad 0 = \left.\frac{\partial U^1}{\partial \theta}\right|_{\theta\uparrow\theta_o} = \left.\frac{\partial U^2}{\partial \theta}\right|_{\theta\downarrow\theta_o} = \int_{C_\rho'} r^{(-2\nu-1)/2}\, \frac{P'_{(2\nu-1)/2}(-\cos\theta_o)P'_{(2\nu-1)/2}(\cos\theta_o)}{W(\nu,\theta_o)}\, D^-(\nu)d\nu.$$

Now in Section 3 we proved that

$$(4.10) \qquad K(\nu,\theta_o) = \frac{P'_{(2\nu-1)/2}(\cos\theta_o)P'_{(2\nu-1)/2}(-\cos\theta_o)}{W(\nu,\theta_o)}$$

could be expressed as the product $K^+(\nu,\theta_o)K^-(\nu,\theta_o)$ of two functions $K^+(\nu,\theta_o)$ and $K^-(\nu,\theta_o)$ which satisfy the equation

$$K^-(\nu,\theta_o) = K^+(-\nu,\theta_o) \tag{4.11}$$

and enjoy the following properties [see Eq. (3.21)]

(4.12)

(a) $K^+(\nu,\theta_o)$ is regular and zeroless in the half plane Re $\nu > -1/2$,

(a') $K^-(\nu,\theta_o)$ is regular and zeroless in the half plane Re $\nu < 1/2$,

(b) $K^+(\nu,\theta_o) \sim k^+(\theta_o, \chi_o)\nu^{1/2}$ as $|\nu + \frac{1}{2}| \to \infty$ in the angular regions $|\arg(\nu + \frac{1}{2})| \leq \frac{\pi}{2}$,

(b') $K^-(\nu,\theta_o) \sim k^+(\theta_o, \chi_o)(-\nu)^{1/2}$ as $|\nu + \frac{1}{2}| \to \infty$ in the angular region $\pi/2 \leq \arg(\frac{1}{2} - \nu) \leq 3\pi/2$ where $k^-(\theta_o \quad _o) = k^+(\theta_o, \quad _o)$ and where $\chi_o = \pi - \theta_o$.

If we now set

$$D^-(\nu) = \frac{1}{(\nu + \frac{1}{2})K^-(\nu,\theta_o)} \tag{4.13}$$

then $D^-(\nu)$ is clearly regular to the left of the contour C_ρ' (since C_ρ' is indented to the left) and vanishes properly as $|\nu| \to \infty$ in this half-plane. Furthermore, on substituting for $D^-(\nu)$ in Eq. (4.9) and employing the fact that $K(\nu,\theta_o) = K^-(\nu,\theta_o)K^+(\nu,\theta_o)$ we get

$$0 = \int_{C_\rho'} \exp\left\{- \left[(\nu + \tfrac{1}{2})\log r\right]\right\} \frac{K^+(\nu,\theta_o)}{(\nu + \frac{1}{2})} d\nu, \qquad r > 1. \tag{4.14}$$

This equation holds when $r > 1$ provided that a): $K^+(\nu,\theta_o)/(\nu + \frac{1}{2})$ is regular to the right of C_ρ', and b): $K^+(\nu,\theta_o)/(\nu + \frac{1}{2}) \sim d|\nu|^{-q}$, $q > 0$, as $|\nu + \frac{1}{2}| \to 0$ in the angular region $|\arg(\nu + \frac{1}{2})| \leq \frac{\pi}{2}$. From Eq. (4.12b) it is clear that condition b) is satisfied. That condition a) is satisfied follows from an inspection of the explicit form of $K^+(\nu,\theta_o)$ in Eq. (3.22). Note that the

factors $\Pi(\nu,\theta_0)$ and $\Pi(\nu,\chi_0)$ each have simple zeros at $\nu = -1/2$ [see Eq. (3.19)]. One of these is cancelled by the simple pole of the factor $\Gamma(\nu + \frac{1}{2})$. Thus $K^+(\nu,\theta_0)$ has a simple zero at $\nu = -1/2$. Since $K^+(\nu,\theta_0)$ is regular to the right of C_ρ' it follows that $K^+(\nu,\theta_0)/(\nu + \frac{1}{2})$ is regular in this region too. The role of the factor $(\nu + \frac{1}{2})^{-1}$ in the expression for $D^-(\nu)$ in Eq. (4.13) is clear; it enables us to satisfy condition b) above. Moreover, as will be seen later, its presence in the integral representations leads automatically to the required behavior of the solution at infinity.

If we substitute for $D^-(\nu)$ in Eqs. (4.6) and (4.6') we get, finally,

$$(4.15)\quad U^1(r,\theta)= \frac{N}{2\pi i}\int_{C_\rho'} r^{(-2\nu-1)/2}\,\frac{P'_{(2\nu-1)/2}(-\cos\theta_0)P_{(2\nu-1)/2}(\cos\theta)}{(\nu + \frac{1}{2})W(\nu,\theta_0)K^-(\nu,\theta_0)}\,d\nu,\quad r \geq 0,\quad 0 \leq \theta \leq \theta_0$$

$$(4.15')\quad U^2(r,\theta)= \frac{N}{2\pi i}\int_{C_\rho'} r^{(-2\nu-1)/2}\,\frac{P'_{(2\nu-1)/2}(\cos\theta_0)P_{(2\nu-1)/2}(-\cos\theta)}{(\nu + \frac{1}{2})W(\nu,\theta_0)K^-(\nu,\theta_0)}\,d\nu,\quad r \geq 0,\quad \theta_0 \leq \theta \leq \pi.$$

The next step is to prove that the above expressions for U^1 and U^2 do indeed provide a solution of our problem. In the following we give a list of results [a) through g) below] which together constitute a proof of the fact that U^1 and U^2 yield a solution of the problem and which at the same time provide a justification for evaluating the representations of U^1 and U^2 and their derivatives by residues. The proofs of the statements given in a) - g) will be found in Appendix II.

In stating these results it is convenient to make the following transformation:

$$(4.16)\qquad \alpha = \nu + \frac{1}{2}\,.$$

Equations (4.15) and (4.15') then become

$$(4.17)\quad U^1(r,\theta) = \frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha}\,\frac{P'_{\alpha-1}(-\cos\theta_o)\,P_{\alpha-1}(\cos\theta)}{\alpha W(\alpha-\frac{1}{2},\theta_o)K^-(\alpha-\frac{1}{2},\theta_o)}\,d\alpha,\quad r \geq 0,\ 0 \leq \theta \leq \theta_o$$

$$(4.17')\quad U^2(r,\theta) = \frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha}\,\frac{P'_{\alpha-1}(\cos\theta)P_{\alpha-1}(\cos\chi)}{\alpha W(\alpha-\frac{1}{2},\theta_o)K^-(\alpha-\frac{1}{2},\theta_o)}\,d\alpha,\quad r \geq 0,\ \theta_o \leq \theta \leq \pi,$$

where C'_ρ in the ν-plane goes over into C_ρ in the α-plane. Except for a small semi-circular arc which bypasses $\alpha = 0$ in a clockwise fashion C_ρ is the imaginary α-axis [see Fig. 3 below].

Then we can prove the following:

a) The integrals in Eq. (4.15) and (4.15') converge uniformly for r and θ in the ranges $0 \leq r \leq R < \infty$, $0 \leq \theta \leq \theta_o$ and $0 \leq r \leq R < \infty$, $\theta_o \leq \theta \leq \pi$ respectively. In particular, therefore,

$$(4.18)\quad \begin{aligned} U^1(r,\theta)\Big|_{\theta\uparrow\theta_o} &= \frac{N}{2\pi i}\int_C r^{-\alpha}\, f^1(\alpha)P_{\alpha-1}(\cos\theta_o)d\alpha \\ U^2(r,\theta)\Big|_{\theta\downarrow\theta_o} &= \frac{N}{2\pi i}\int_C r^{-\alpha}\, f^2(\alpha)P_{\alpha-1}(\cos\chi_o)d\alpha, \end{aligned}$$

where $f^1(\alpha)$ and $f^2(\alpha)$ are defined by

$$(4.19)\quad \begin{aligned} f^1(\alpha) &= \frac{P'_{\alpha-1}(\cos\chi_o)}{\alpha W(\alpha-\frac{1}{2},\theta_o)K^-(\alpha-\frac{1}{2},\theta_o)}\ , \\ f^2(\alpha) &= \frac{P'_{\alpha-1}(\cos\theta_o)}{\alpha W(\alpha-\frac{1}{2},\theta_o)K^-(\alpha-\frac{1}{2},\theta_o)}\ . \end{aligned}$$

b) On setting $\alpha = |\alpha|e^{i\beta}$, the contributions of the integrands of Eq. (4.17 and Eq. (4.17') when integrated over an appropriate sequence of the circular arcs $C_+^{|\alpha_o^n|}$ and $C_-^{|\alpha_o^n|}$ [see Fig. 3] approach zero with $|\alpha_o^n|^{-1}$

uniformly in β for β in the ranges $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \beta \leq \frac{3}{2}\pi$, respectively. Consequently the integrals in Eqs. (4.17) and (4.17') may be evaluated by residues.

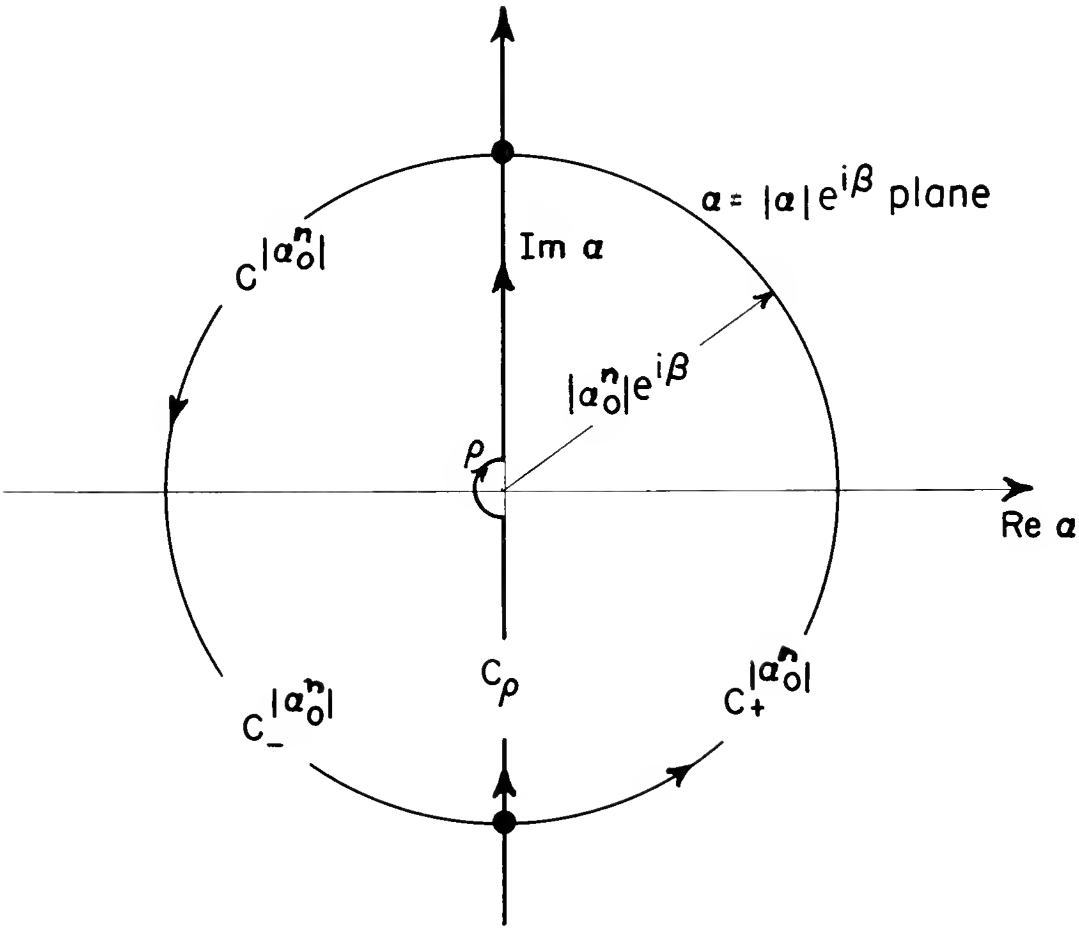

Figure 3

c) In the region $0 \leq r \leq R < \infty$, $0 \leq \theta \leq \theta_0 - \varepsilon$, $\varepsilon > 0$, the integrals

$$\int_{C_\rho} f^1(\alpha) \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} r^{-\alpha} P_{\alpha-1}(\cos\theta)\, d\alpha,$$

(4.20)

$$\int_{C_\rho} f^1(\alpha) \begin{bmatrix} \frac{\partial^2}{\partial r^2} \\ \frac{\partial^2}{\partial \theta^2} \end{bmatrix} r^{-\alpha} P_{\alpha-1}(\cos\theta)\, d\alpha$$

are uniformly convergent, and also in the region

$0 \leq r \leq R < \infty$, $\theta_0 + \delta \leq \theta \leq \pi$, $\delta > 0$, the integrals

$$\int_{C_\rho} f^2(\alpha) \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} r^{-\alpha} P_{\alpha-1}(\cos \pi - \theta)\, d\alpha,$$

(4.21)

$$\int_{C_\rho} f^2(\alpha) \begin{bmatrix} \frac{\partial^2}{\partial r^2} \\ \frac{\partial^2}{\partial \theta^2} \end{bmatrix} r^{-\alpha} P_{\alpha-1}(\cos \pi - \theta)\, d\alpha$$

are uniformly convergent. Consequently the operators $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial^2}{\partial \theta^2}$, $\frac{\partial^2}{\partial \theta^2}$ commute with the integral operator in Eqs. (4.17) and (4.17'). In particular these integral expressions are continuous across $r = 1$. Moreover Laplace's equation is satisfied in the stated regions of validity.

By suitably deforming the contour C_ρ it is possible to obtain integral representations which are equivalent to those mentioned in Eqs. (4.17) and (4.17') and which, provided r is bounded away from unity, converge for all θ in the ranges $0 \leq \theta \leq \theta_0$, $\theta_0 \leq \theta \leq \pi$. In the following we discuss briefly these equivalent representations and mention some properties which will be useful later.

d) In Eqs. (4.17) and (4.17'), for $r \geq 1$ and $r \leq 1$, we may deform the contour C_ρ to the contours $C^+_{\pi/4}$ and $C^-_{\pi/4}$ respectively [see Fig. 4]. [In Fig. 4 the extensions of the linear parts of $C^+_{\pi/4}$ and $C^-_{\pi/4}$ are chosen to go through the origin. The 90-degree arc in the left-half-plane which is included between the linear portions of $C^-_{\pi/4}$ is shared by the contour $C^+_{\pi/4}$.] Therefore we may write

$$(4.22)\quad \begin{bmatrix} U^1 \\ U^2 \end{bmatrix} = \frac{N}{2\pi i} \int_{C^+_{\pi/4}} r^{-\alpha} \begin{bmatrix} f^1(\alpha)P_{\alpha-1}(\cos\theta) \\ f^2(\alpha)P_{\alpha-1}(\cos\pi-\theta) \end{bmatrix} d\alpha,\ r \geq 1, \begin{bmatrix} 0 \leq \theta \leq \theta_o \\ \theta_o \leq \theta \leq \pi \end{bmatrix}$$

$$(4.23)\quad \begin{bmatrix} U^1 \\ U^2 \end{bmatrix} = \frac{N}{2\pi i} \int_{C^-_{\pi/4}} r^{-\alpha} \begin{bmatrix} f^1(\alpha)P_{\alpha-1}(\cos\theta) \\ f^2(\alpha)P_{\alpha-1}(\cos\pi-\theta) \end{bmatrix} d\alpha,\ r \leq 1, \begin{bmatrix} 0 \leq \theta \leq \theta_o \\ \theta_o \leq \theta \leq \pi \end{bmatrix}.$$

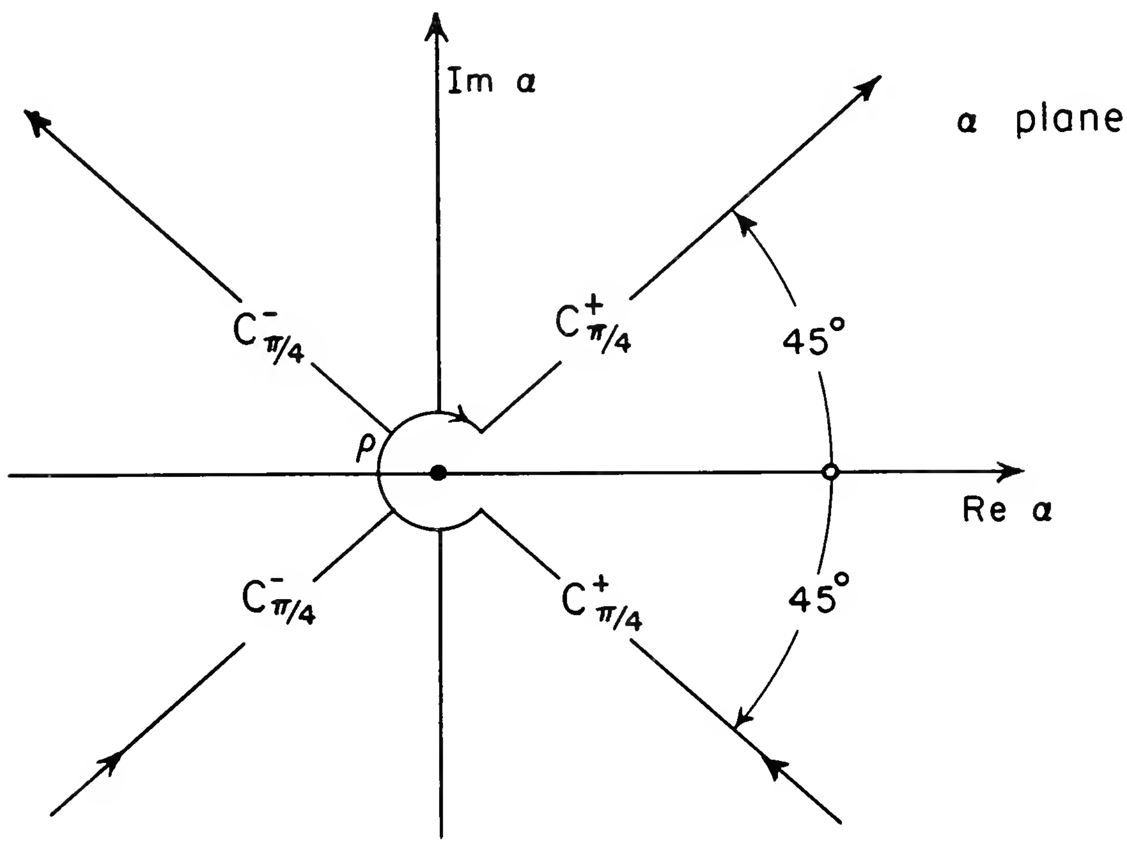

Figure 4

e) When $1 + \varepsilon \leq r \leq R < \infty$, $\varepsilon > 0$, and $0 \leq \theta \leq \theta_o$ or $\theta_o \leq \theta \leq \pi$ in Eq. (4.22) we may pass under the integral signs with the operators $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial r^2}$, $\frac{\partial}{\partial\theta}$ or $\frac{\partial^2}{\partial\theta}$ and still obtain integral expressions that converge uniformly with respect to these variables when they are allowed to vary in the indicated

ranges. The same is true of the integrals in Eq. (4.23) when r is confined to the closed interval $0 \leq r \leq 1 - \varepsilon$ and θ in the closed intervals $0 \leq \theta \leq \theta_o$ and $\theta_o \leq \theta \leq \pi$. In particular, we may write

$$(4.24)\qquad \begin{bmatrix} \left.\dfrac{\partial U^1(r,\theta)}{\partial\theta}\right|_{\theta\uparrow\theta_o} \\ \\ \left.\dfrac{\partial U^2(r,\theta)}{\partial\theta}\right|_{\theta\downarrow\theta_o} \end{bmatrix} = \frac{N}{2\pi i}\int_{C^+_{\pi/4}} r^{-\alpha} \begin{bmatrix} f^1(\alpha)P'_{\alpha-1}(\cos\theta_o) \\ \\ f^2(\alpha)P'_{\alpha-1}(\cos\pi-\theta_o) \end{bmatrix} d\alpha, \qquad r > 1,$$

$$(4.25)\qquad \begin{bmatrix} \left.\dfrac{\partial U^1(r,\theta)}{\partial\theta}\right|_{\theta\uparrow\theta_o} \\ \\ \left.\dfrac{\partial U^2(r,\theta)}{\partial\theta}\right|_{\theta\downarrow\theta_o} \end{bmatrix} = \frac{N}{2\pi i}\int_{C^-_{\pi/4}} r^{-\alpha} \begin{bmatrix} f^1(\alpha)P'_{\alpha-1}(\cos\theta_o) \\ \\ f^2(\alpha)P'_{\alpha-1}(\cos\pi-\theta_o) \end{bmatrix} d\alpha, \qquad r < 1.$$

f) Consider the integrals derived from (4.22) and (4.23) by differentiating under the integral signs with respect to θ and r. The values obtained by integrating along the contours $C_+^{|\alpha_o^n|}$ and $C_-^{|\alpha_o^n|}$ approach zero with $|\alpha_o^n|^{-1}$ if an appropriate sequence of $|\alpha_o^n|$, n = 1, 2, ··· is chosen. Therefore we may evaluate these integrals by residues. This means that we may differentiate the eigenseries expansions of U^1 and U^2 term by term.

g) Finally, for $0 \leq r < 1$ the integral representations of $\dfrac{\partial U^i}{\partial r}$, $\dfrac{\partial U^i}{\partial\theta}$, $\dfrac{\partial U^i}{\partial r^2}$ and $\dfrac{\partial U^i}{\partial\theta^2}$, i = 1,2, are continuous across $\theta = \theta_o$ so that Laplace's equation is satisfied for all r, $0 \leq r < 1$, $0 \leq \theta \leq 2\pi$.

As already mentioned, the proof of these assertions will be given in Appendix II. It suffices to say here that these facts follow from the known asymptotic behavior of $K^+(\alpha - \frac{1}{2}, \theta_o)$, $K^-(\alpha - \frac{1}{2}, \theta_o)$ and of $P_{\alpha-1}(\pm\cos\theta)$ and $P'_{\alpha-1}(\pm\cos\theta)$ in the complex α-plane.

Before giving the eigenseries expansions of U^1 and U^2 it will be useful

to have at our disposal alternative forms of Eqs. (4.17) and (4.17'). These are obtained by solving for $K^-(\alpha - \frac{1}{2}, \theta_o)$ in Eq. (3.13) and substituting the resulting expression into the integrands of Eqs. (4.17) and (4.17'). The resulting integral representations of U^1 and U^2 are

$$(4.26)\quad U^1(r,\theta) = \frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha}\, \frac{P_{\alpha-1}(-\cos\theta)}{\alpha P'_{\alpha-1}(\cos\theta_o)}\, K^+(\alpha - \tfrac{1}{2}, \theta_o)\,d\alpha, \quad r \geq 0, \quad 0 \leq \theta \leq \theta_o,$$

$$(4.26')\quad U^2(r,\theta) = \frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha}\, \frac{P_{\alpha-1}(-\cos\theta)}{\alpha P'_{\alpha-1}(-\cos\theta_o)}\, K^+(\alpha - \tfrac{1}{2}, \theta_o)\,d\alpha, \quad r \geq 0, \quad \theta_o \leq \theta \leq \pi.$$

We shall employ Eqs. (4.17) and (4.17') to obtain eigenseries expansions in the region $r \leq 1$ and Eqs. (4.26) and (4.26') to obtain eigenseries expansions in the region $r \geq 1$. The virtue of this procedure is that we avoid the appearance of terms involving $\frac{\partial K^+(\alpha - \frac{1}{2}, \theta)}{\partial \alpha}$ and $\frac{\partial K^-(\alpha - \frac{1}{2}, \theta_o)}{\partial \alpha}$ about which we do not have much ready information. Actually, it is necessary to evaluate only the expression for $U^1(r,\theta)$ by residues, as we shall now show. Let us write

$$(4.27)\qquad U^i(r,\theta) = U^i(r,\theta,\theta_o) \qquad i = 1, 2$$

to exhibit the θ_o-dependence of U^1 and U^2. Now if in Eq. (4.26') we write $-\cos\theta$ and $-\cos\theta_o$ as $\cos(\pi-\theta)$ and $\cos(\pi-\theta_o)$, and set $\chi = \pi-\theta$ and $\chi_o = \pi-\theta_o$; then it is easy to see that

$$(4.28)'\quad U^2(r,\theta,\theta_o) = -\frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha}\, \frac{P_{\alpha-1}(\cos\chi)K^+(\alpha - \frac{1}{2}, \theta_o)}{\left.\frac{dP_{\alpha-1}(\cos\chi)}{d\chi}\right|_{\chi=\chi_o}}\,d\alpha = -U^1(r, \chi, \chi_o).$$

This result follows from the fact that $K^+(\alpha - \frac{1}{2}, \theta_o) = K^+(\alpha - \frac{1}{2}, \chi_o)$ [cf. Eq. (3.22)] and the fact that

$$P'_{\alpha-1}(-\cos\theta_0) = \left.\frac{dP_{\alpha-1}(-\cos\theta)}{d\theta}\right|_{\theta = \theta_0} = \left.\frac{dP_{\alpha-1}(\cos\chi)}{d\chi}\right|_{\chi = \chi_0}.$$

Therefore if we obtain the eigenfunction representation for $U^1(r,\theta,\theta_0)$ in say the region $r \geq 1$, $0 \leq \theta \leq \theta_0$, the representation for $U^2(r,\theta,\theta_0)$ in the region $r \geq 1$, $\theta_0 \leq \theta \leq \pi$ may be obtained by setting $U^2(r,\theta,\theta_0) = -U^1(r,\chi,\chi_0)$.

5. Eigenfunction Expansions - Conductivity of the Opening

5.1 Expansions in the Region $r \geq 1$; Determination of the Normalization Factor N

To find the eigenfunction expansions for $U^1(r,\theta;\theta_0)$ in the region $r \geq 1$, $0 \leq \theta \leq \theta_0$ we employ, for reasons mentioned above, the integral representation for U^1 given in Eq. (4.26). Since $(K^+(\alpha - \frac{1}{2}, \theta_0)/\alpha)$ is regular to the right of the contour C_ρ [see p. 20] we see that the only poles in the integrand of Eq. (3.20) which lie to the right of C_ρ are located at the zeros of $P'_{\alpha-1}(\cos\theta_0)$ which lie to the right of C_ρ. These zeros are given, in virtue of Eq. (3.12), by

$$(5.1) \qquad \alpha_0(\theta_0) = 0; \quad \alpha_p(\theta_0) = \frac{1}{2} + \frac{\pi}{\theta_0}\left(p - \frac{3}{4}\right) + O\left(\frac{1}{p}\right), \ p = 1,2,\cdots,$$

where $\alpha_1(\theta_0) = 1$ for all θ_0 in the range $0 < \theta_0 < \pi$, and where in particular $\alpha_p = \frac{1}{2} + 2(p - \frac{3}{4})$, $p = 1, 2, \cdots$, for $\theta_0 = \frac{\pi}{2}$, [cf. Eq. (3.11)].

Expanding the right-hand side of Eq. (4.26) by residues [see b, p.22] we get

$$(5.2) \quad U^1(r,\theta;\theta_0) = -Na_0^1(\theta_0) - N\sum_{p=1}^{\infty} a_p^1(\theta_0)P_{\alpha_p(\theta_0)-1}(\cos\theta)r^{-\alpha_p(\theta_0)}, \ r \geq 1, \ 0 \leq \theta \leq \theta_0,$$

where

$$(5.3)\quad a_p^1(\theta_c) = \lim_{\alpha\to\alpha_p(\theta_o)} \left\{ \frac{[\alpha-\alpha_p(\theta_o)]K^+(\alpha-\frac{1}{2},\theta_c)}{\alpha P'_{\alpha-1}(\cos\theta_c)} \right\} = \frac{K_p^+(\theta_o)}{\left[\frac{\partial^2 P_{\nu-1}(\cos\theta)}{\partial\nu\partial\theta}\right]\Bigg|_{\substack{\theta=\theta_o\\ \nu=\alpha_p(\theta_o)}}},$$

$K_p^+(\theta_c)$ being defined as follows:*

$$K_p^+(\theta_c) = \lim_{\alpha\to\alpha_p(\theta_o)} \left\{ \frac{K^+(\alpha-\frac{1}{2},\theta_o)}{\alpha} \right\}$$

$$(5.4)\qquad = 2\sqrt{2\sin\theta_c\, P'_{-1/2}(\cos\theta_o)P'_{-1/2}(\cos\chi_o)}\,\exp\left\{\left(\alpha_p(\theta_o)-\frac{1}{2}\right)\left(t-\frac{1}{4}\right)\right\}\Gamma\left[\alpha_p(\theta_c)+1\right]$$

$$\cdot\prod_{\ell=2}^{\infty}\left\{\left[1+\left(\frac{\alpha_p(\theta_o)-\frac{1}{2}}{\nu_\ell(\theta_o)}\right)\right]\exp\left(\frac{\frac{1}{2}-\alpha_p(\theta_o)}{\nu_\ell(\theta_o)}\right)\right\}$$

$$\cdot\prod_{\ell=2}^{\infty}\left\{\left[1+\left(\frac{\alpha_p(\theta_o)-\frac{1}{2}}{\nu_\ell(\chi_o)}\right)\right]\exp\left(\frac{\frac{1}{2}-\alpha_p(\theta_o)}{\nu_\ell(\chi_o)}\right)\right\}\qquad p=0,1,2,\cdots.$$

The expressicns for the first two coefficients, $a_o^1(\theta_c)$ and $a_1^1(\theta_o)$, can be recast in a simpler form. To effect this simplification we employ the formulas

$$(5.5)\qquad P'_{\nu-1}(\cos\theta_o) = -\nu(\nu-1)P^{-1}_{\nu-1}(\cos\theta_o),$$

[cf. Eq. (3.6)] and

$$(5.6)\qquad P_o^{-1}(\cos\theta_o) = P_{-1}^{-1}(\cos\theta_o) = \tan\frac{\theta_o}{2}$$

[cf. [2], p.63] which in conjunction with the first equation in Eq. (5.3) yield,

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* It should be observed that $K_p(\theta_o)$, p=0,1,2,... is positive imaginary. This follows from the fact that $K(\alpha-\frac{1}{2},\theta_o)$ is positive imaginary when α is real [see p. 14] and the fact that $\alpha_p(\theta_o)$ is real.

after the proper limiting operations have been performed,

$$a_o^1(\theta_o) = K_o^+(\theta_o)\mathrm{ctn}(\theta_o/2)$$

(5.7)

$$a_1^1(\theta_o) = -K_1^+(\theta_o)\mathrm{ctn}(\theta_o/2).$$

Employing the preceding results in conjunction with Eq. (4.28) we have, in the region $r \geq 1$, $0 \leq \chi \leq \chi_o$,

$$(5.8)\quad U^2(r,\theta,\theta_o) = N\, a_o^1(\chi_o) + N \sum_{p=1}^{\infty} a_p^1(\chi_o) P_{\alpha_p(\chi_o)-1}(\cos\chi) r^{-\alpha_p(\chi_o)}, \quad r \geq 1,$$

$$\theta_o \leq \theta \leq \pi,$$

where $\chi = \pi - \theta$, and $\alpha_p(\chi_o) = \frac{1}{2} + \frac{\pi}{\chi_o}(p - \frac{3}{4}) + O(\frac{1}{p})$, and where

$$a_p^1(\chi_o) = \frac{K_p^+(\chi_o)}{\left[\dfrac{\partial^2 P_{\alpha-1}(\cos\chi)}{\partial\alpha\partial\chi}\right]\Bigg|_{\substack{\chi = \chi_o \\ \alpha = \alpha_p(\chi_o)}}}, \qquad p = 2,3,4,\ \ldots ,$$

$$(5.9)\qquad a_o^1(\chi_o) = K_o^+(\chi_o)\mathrm{ctn}(\chi_o/2) = K_o^+(\chi_o)\tan(\theta_o/2),$$

$$a_1^1(\chi_o) = K_o^+(\chi_o)\mathrm{ctn}(\chi_o/2) = -K_1^+(\chi_o)\tan(\theta_o/2) \quad .$$

In particular at $\theta_o = \chi_o = \frac{\pi}{2}$ we have [cf. Eq. (3.37)]

$$(5.10)\qquad \frac{K^+(\alpha - \frac{1}{2}, \frac{\pi}{2})}{\alpha} = i\,\frac{\Gamma(\frac{1}{2} + \frac{\alpha}{2})}{\alpha\Gamma(\frac{\alpha}{2})} = i\,\frac{\Gamma(\frac{1}{2} + \frac{\alpha}{2})}{2\Gamma(\frac{\alpha}{2} + 1)} \; .$$

Also

(5.11) $$P'_{\alpha-1}(0) = \frac{2\sqrt{\pi}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2})}$$

[cf. Eq. (3.10)].

Employing these formulas in conjunction with Eq. (5.3) and the fact that $\Gamma(\frac{1}{2}+\frac{\alpha}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2}) = \pi/\cos(\frac{\pi\alpha}{2})$, we get

$$a_p^1(\tfrac{\pi}{2}) = i \lim_{\alpha \to \alpha_p(\frac{\pi}{2})} \left\{ \frac{\left(\alpha-\alpha_p(\frac{\pi}{2})\right)\Gamma(\frac{1}{2}+\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{2\Gamma(\frac{\alpha}{2}+1)\, 2\sqrt{\pi}} \right\}$$

$$= i \lim_{\alpha \to \alpha_p(\frac{\pi}{2})} \left\{ \frac{\left(\alpha-\alpha_p(\frac{\pi}{2})\right)\sqrt{\pi}}{2\alpha\cos(\frac{\pi\alpha}{2})} \right\}.$$

Since $\alpha_p = 2p - 1$, $p = 1,2,\cdots$ it follows that at $\theta_0 = \frac{\pi}{2}$,

(5.12) $$a_p^1(\tfrac{\pi}{2}) = -\frac{i}{\sqrt{\pi}(2p-1)\sin(\frac{2p-1}{2})\pi} = i\,\frac{(-1)^p}{\sqrt{\pi}\,(2p-1)} \qquad p = 1,2,\cdots.$$

For $p = 0$ it is easy to prove from Eqs. (5.7) and (5.10) that

(5.12') $$a_0^1(\tfrac{\pi}{2}) = i\,\frac{\sqrt{\pi}}{2}.$$

Eigenfunction representations for the derivatives of $U^1(r,\theta;\theta_0)$ and $U^2(r,\theta;\theta_0)$ can be obtained from Eqs. (5.2) and (5.8) by termwise differentiations [see p. 26 f)]. In particular, for $r \gg 1$

(5.13)
$$\frac{\partial U^1(r,\theta,\theta_0)}{\partial r} \sim \frac{Na_1^1(\theta_0)}{r^2} + \cdots, \qquad 0 \leq \theta \leq \theta_0$$

$$\frac{\partial U^2(r,\theta,\theta_0)}{\partial r} \sim -\frac{Na_1^1(\chi_0)}{r^2} + \cdots, \qquad \theta_0 \leq \theta \leq \pi.$$

In order that the incoming flux in the region $0 \leq \theta \leq \theta_0$ be f [cf. paragraph following Eq. (2.4d)] we must have

(5.14) $$N = -\frac{f}{4\pi \sin^2(\Theta_o/2)\, a_1^1(\Theta_o)} ,$$

which in conjunction with Eq. (5.7) yields

(5.14') $$N = \frac{f}{2\pi \sin\Theta_o K_1^+(\Theta_o)} .$$

Note that N is a negative imaginary quantity since $K_1^+(\Theta_o)$ is a positive imaginary quantity [see footnote on p. 29]. It follows that the quantities $Na_p^1(\Theta_o)$ and $Na_p^2(\chi_o)$ appearing in Eqs. (5.2) and (5.8) are real.

When $\Theta_o = \pi/2$, $a_1^1(\frac{\pi}{2}) = -i/\sqrt{\pi}$, from Eq. (5.12). Employing Eq. (5.14) we have

(5.15) $$N(\pi/2) = f/(2i\sqrt{\pi}).$$

Since $\alpha_p(\pi/2) = 0$ and $\alpha_p(\pi/2) = 2p-1$, $p = 1,2,\cdots$, we find, using (5.12), (5.13) and (5.15) that the eigenfunction expansions given in Eqs. (5.2) and (5.8) become

(5.16)
$$U^1 = -\frac{f}{4} - \frac{f}{2\pi}\sum_{p=1}^{\infty} \frac{(-1)^p}{(2p-1)} P_{2p-2}(\cos\Theta) r^{-(2p-1)}, \qquad 0 \leq \Theta \leq \pi/2$$
$$U^2 = \frac{f}{4} + \frac{f}{2\pi}\sum_{p=1}^{\infty} \frac{(-1)^p}{(2p-1)} P_{2p-2}(\cos\Theta) r^{-(2p-1)}, \qquad \pi/2 \leq \Theta \leq \pi$$

where we have employed the fact that $P_{2p-2}(-\cos\Theta) = P_{2p-2}(\cos\Theta)$ in the expression for U^2. Clearly $U^1 = -U^2$ when $\Theta = \pi/2$ and $r > 1$ and hence the potential is discontinuous across the cone, which, since $\Theta_o = \pi/2$, is a plane screen. The normal derivatives $\partial U^1/\partial\Theta$ and $\partial U^2/\partial\Theta$ both vanish on the screen

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* In particular since $K_o(\Theta)$ and $K_o(\chi_o)$ are positive imaginary quantities [see the two preceeding footnotes], $a_o^1(\Theta_o)$ and $a_o^1(\chi_o)$ are positive imaginary quantities [cf. Eqs. (5.7) and (5.9)]. Hence $Na_o^1(\chi_o)$ and $Na_o^1(\Theta_o)$ are positive quantities.

because $P_{2p-2}(\cos\theta)$, $p = 1,2$, is an even function of $\cos\theta$ and as a result

$$\frac{\partial P_{2p-2}(\cos\theta)}{\partial\theta}\bigg|_{\theta = \pi/2} = 0.$$

It is of interest to determine the flux at infinity in the region $\theta_o \leq \theta \leq \pi$. In this region we have in virtue of Eq. (5.13) and (5.9)

$$\frac{\partial U^2(r,\theta;\theta_o)}{\partial r} \sim \frac{f}{4\pi \cos^2 \frac{\theta_o}{2}} \quad \frac{1}{r^2} \qquad r \to \infty$$

where we have made use of the fact that*

(5.17) $$K_1^+(\chi_o) = K_1^+(\theta_o).$$

It is now an easy matter to verify that in the region $\theta_o \leq \theta \leq \pi$ the flux at infinity is outgoing and has the magnitude f. Incidentally, we have proved the fact that the flux entering at $r = \infty$ in the angular region $0 \leq \theta < \theta_o$ is equal to the flux leaving at $r = \infty$ in the angular region $\theta_o \leq \theta \leq \pi$.

5.2 Expansions in the Region $r \leq 1$

In the region $r \leq 1$, as already noted, it is convenient to use the integral representations given in Eqs. (4.17) and (4.17'). By an inspection of the integrands we see that the only poles in these integrands to the left of the contour C_ρ come from the negative zeros of $W(\alpha - \frac{1}{2},\theta_o)$. From Eq. (3.4)

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* This result follows immediately from Eq. (5.4) if we recall that $\alpha_1(\theta_o) = \alpha_1(\chi_o) = 1$. For similar reasons we also have $K_o^+(\chi_o) = K_o^+(\theta_o)$.

(5.18) $$W(\alpha - \tfrac{1}{2}, \theta_o) = \frac{2 \sin \pi \alpha}{\pi \sin \theta_o},$$

so that the poles of the integrands, $\alpha_{-p}(\theta_o)$, are given by

$$\alpha_{-p}(\theta_o) = -p, \qquad p = 1, 2, \cdots.$$

Expanding these integrals by residues [see p. 22, b)] and using the fact that $P_{-p-1}(\cos\theta) = P_p(\cos\theta)$ we get

(5.19)
$$U^1(r,\theta;\theta_o) = N\left[\sum_{p=1}^{\infty} (-1)^p\, b_p^1(\theta_o) P_p(\cos\theta) r^p\right], \quad 0 \le r < 1, \quad 0 \le \theta \le \theta_o,$$

$$U^2(r,\theta;\theta_o) = N\left[\sum_{p=1}^{\infty} (-1)^p\, b_p^2(\theta_o) P_p(-\cos\theta) r^p\right], \quad 0 \le r < 1, \quad \theta_o \le \theta \le \pi,$$

where

(5.20)
$$b_p^1(\theta_o) = \frac{\sin\theta_o \left[\left.\frac{dP_p(-\cos\theta)}{d\theta}\right|_{\theta=\theta_o}\right]}{2\, K^-(-p - \frac{1}{2}, \theta_o)},$$

$$b_p^2(\theta_o) = \frac{\sin\theta_o \left[\left.\frac{dP_p(\cos\theta)}{d\theta}\right|_{\theta=\theta_o}\right]}{2\, K^-(-p - \frac{1}{2}, \theta_o)}, \qquad p = 1, 2, \cdots.$$

Employing the fact that $P_p(-\cos\theta) = (-1)^p P_p(\cos\theta)$, it is easy to verify that

$$U^1(r,\theta;\theta_o) = U^2(r,\theta;\theta_o) = N\left[\sum_{p=1}^{\infty} b_p(\theta_o) P_p(\cos\theta) r^p\right], \quad 0 \le r \le 1,\ 0 \le \theta \le \pi,$$

where*

(5.21) $$b_p(\theta_o) = \frac{\sin\theta_o \left[\left.\frac{dP_p(\cos\theta)}{d\theta}\right|_{\theta=\theta_o}\right]}{2K^+(p + \frac{1}{2}, \theta_o)}.$$

---

* From Eq. (3.21c) we know that $K^-(\nu,\theta_o) = K^+(-\nu,\theta_o)$. It follows that $K^-(-p - \frac{1}{2}, \theta_o) = K^+(p + \frac{1}{2}, \theta_o)$.

When $\Theta_o = \pi/2 = \pi - \Theta_o$ we have from Eqs. (5.10) and (5.11)

$$b_p(\tfrac{\pi}{2}) = \frac{1}{2} \left. \frac{P'_{(2\nu-1)/2}(0)}{K^+(\nu, \frac{\pi}{2})} \right|_{\nu = p + \frac{1}{2}} \qquad p = 1,2,\cdots,$$

$$= \left. \frac{\sqrt{\pi}}{i\Gamma(\frac{1}{4} - \frac{\nu}{2})\Gamma(\frac{3}{4} + \frac{\nu}{2})} \right|_{\nu = p + \frac{1}{2}} \qquad \text{[from Eqs. (3.10) and (3.37)]}$$

$$= \frac{\sqrt{\pi}}{i\Gamma(-\frac{p}{2})\Gamma(1 + \frac{p}{2})} = \frac{-\sin(p\pi/2)}{i\sqrt{\pi}},$$

since $\sin \pi z = \pi / \Gamma(z)\Gamma(1-z)$. It follows that

$$(5.22) \qquad b_p(\tfrac{\pi}{2}) = \begin{cases} 0, & l = 1,2, \cdots, \\ \dfrac{(-1)^{2k-1}}{i\sqrt{\pi}}, & p = 2k-1, \quad k = 1,2,\cdots. \end{cases}$$

The velocity potential $U(r,\Theta)$ in the region $0 \leq r < 1$ is therefore given by

$$(5.23) \qquad U(r,\Theta) = \frac{N}{i\sqrt{\pi}} \left[\sum_{k=1}^{\infty} (-1)^{2k-1} P_{2k-1}(\cos\Theta) r^{2k-1} \right].$$

Note that the right-hand side of (5.23) is an odd function of $\cos \Theta$, and that N is positive imaginary [see Eqs. (5.10) and (5.16)].

5.3 Conductivity of the Opening

Let U_o'' and U_o' be the constant potentials at infinity in the regions where the flux is outgoing and incoming respectively. Then the conductivity of the opening $\sigma(\Theta_o)$ is defined as follows:

$$(5.24) \qquad \sigma(\Theta_o) = f \Big/ (U'' - U').$$

In the present problem flux is incoming in the region $0 \leq \Theta \leq \Theta_o$ and outgoing in the region $\Theta_o \leq \Theta \leq \pi$. Let

$$(5.25) \qquad U_o^1 = \lim_{r \to 0} U^1(r,\Theta),\ 0 \leq \Theta \leq \Theta_o; \quad U_o^2 = \lim U^2(r,\Theta),\ \Theta_o \leq \Theta \leq \pi,$$

[cf. Eq. (2.4d)]. Then we have [see Eqs. (5.2) and (5.8)]

$$U'' = Na_0^2(\chi_0), \qquad U' = -Na_0^1(\theta_0). \tag{5.26}$$

It has already been remarked [see footnote p.32] that $Na_0^2(\chi_0)$ and $Na_0^1(\theta_0)$ are positive quantities. The conductivity $\sigma(\theta_0)$ is therefore a positive number. It is also interesting to note that the fluid flows from lower to higher potential. Had we started with incoming flux at infinity in the region $\theta_0 \leq \theta \leq \pi$ the same remarks would be true as can easily be verified by substituting $-f$ for f in the results of the preceeding sections. Moreover the expression for the conductivity for this case would be the same as derived below.

Employing Eq. (5.15) we then have

$$\sigma(\theta_0) = \frac{4\pi \sin^2(\theta_0/2)\; a_1^1(\theta_0)}{\left[a_0^1(\theta_0) + a_0^1(\chi_0)\right]}, \tag{5.27}$$

which in virtue of Eqs. (5.7) and (5.9) becomes

$$\sigma(\theta_0) = \frac{4\pi \sin^2(\theta_0/2)\left[K_1^+(\theta_0)\mathrm{ctn}(\theta_0/2)\right]}{\left[K_0^+(\theta_0)\mathrm{ctn}(\theta_0/2)+K_0^+(\chi_0)\tan(\theta_0/2)\right]}, \tag{5.28}$$

$K_p(\theta_0)$, $p = 0, 1$ being defined in Eq. (5.4). Now, as mentioned above [see footnote p.33] $K_0^+(\theta_0) = K_0^+(\chi_0)$. Therefore we have

$$\sigma(\theta_0) = \frac{\pi\left[K_1^+(\theta_0)\right]\sin^2\theta_0}{\left[K_0^+(\theta_0)\right]}. \tag{5.29}$$

Now it can be shown [Appendix III] that

$$K_0^+(\theta_0)K_1^+(\theta_0) = -\frac{\sin\theta_0}{2}. \tag{5.30}$$

Then clearly

$$\sigma(\theta_0) = -2\pi\left[K_1^+(\theta_0)\right]^2\sin\theta_0, \qquad 0 < \theta_0 < \pi.^* \tag{5.31}$$

---

*It should be recalled that $K_1^+(\theta_0)$ is an imaginary quantity. The expression for $\sigma(\theta_0)$ on the right-hand side of Eq. (5.31) is therefore real and positive.

When $\theta_o = \frac{\pi}{2}$, we have from Eq. (5.10)

(5.32) $$K_\perp^+(\tfrac{\pi}{2}) = \frac{i}{2\Gamma(\frac{3}{2})} = \frac{i}{\sqrt{\pi}} .$$

Thus

(5.33) $$\sigma(\tfrac{\pi}{2}) = 2.$$

This result agrees with the known result for the conductivity of a circular aperture of unit radius in a plane screen, [cf. [5] p . 518, Eq. (19)].*

The function $K_\perp^+(\theta_o)$ in Eq. (5.30) may be expanded in a Taylor's series about the angle $\theta_o = \pi/2$. In this connection it is to be noted that Eq. (5.17) implies that $K_\perp(\theta_o)$ is an even function of $(\theta_o - \pi/2)$. The series expansion for $K_\perp(\theta_o)$, up to terms of $O[(\theta_o - \pi/2)^4]$, may therefore be written as follows:

(5.34) $$K_\perp = K_\perp(\tfrac{\pi}{2})\left\{1 + \frac{1}{2}\left[\frac{1}{K_\perp(\frac{\pi}{2})}\left.\frac{\partial^2 K(\theta_o)}{\partial\theta_o^2}\right|_{\theta_o = \pi/2}\right](\theta_o - \tfrac{\pi}{2})^2\right\} + O\left[(\theta_o - \tfrac{\pi}{2})^4\right],$$

$$0 < \theta_o < \pi.$$

It is not difficult to evaluate $d^2K(\theta_o)/d\theta_o^2$ at $\theta_o = \pi/2$. For the sake of brevity, however, we refrain from giving the details and merely quote the following final result:

$$\left.\frac{d^2K_1(\theta_o)}{d\theta_o^2}\right|_{\theta_o = \frac{\pi}{2}} = K_\perp(\tfrac{\pi}{2})\left\{\frac{3}{4} + \frac{2}{\pi^2} + \frac{1}{\pi^4}\left(\frac{\Gamma(\frac{1}{4})}{2}\right)^8 - \sum_{p=1}^{\infty}\left\{\frac{1}{(2p-\frac{3}{2})^2}\left[\left(\left.\frac{d^2\nu_p(\theta_o)}{d\theta_o^2}\right|_{\theta_o = \frac{\pi}{2}}\right) - \frac{1}{p-\frac{3}{2}}\right]\right\}\right.$$

(5.35) $$\left. - \frac{1}{2}\sum_{p=1}^{\infty}\left[\frac{(6p-\frac{1}{2})}{(2p-\frac{1}{2})^2(2p-\frac{3}{2})^3}\left(\left.\frac{d\nu_p(\theta_o)}{d\theta_o}\right|_{\theta_o = \frac{\pi}{2}}\right) - \frac{1}{(2p-\frac{1}{2})(2p-\frac{3}{2})^2}\left.\frac{d^2\nu_p(\theta_o)}{d^2\theta_o}\right|_{\theta_o = \frac{\pi}{2}}\right]\right\},$$

---

*When the radius of the circular aperture is $b \neq 1$ it is easily verified that $\sigma(\theta_o) = -2\pi b\, K_1^+(\theta_o)^2\sin\theta_o$. In particular, when $\theta = \pi/2$. $\sigma(\theta_o) = 2b$.

where*

$$(5.36)\qquad \left.\frac{d\nu_p}{d\theta_o}\right|_{\theta_o=\frac{\pi}{2}} = \frac{4}{\pi}\left(\frac{\Gamma(\frac{1}{2}+p)}{\Gamma(p)}\right)^2\left(\frac{p-1}{p-\frac{1}{2}}\right),\quad p = 1,2,\cdots,$$

$$(5.37)\qquad \left.\frac{d^2\nu_p}{d\theta_o^2}\right|_{\theta_o=\frac{\pi}{2}} = 2\left(\left.\frac{d\nu_p}{d\theta_o}\right|_{\theta_o=\frac{\pi}{2}}\right)^2\left[\Psi(p-1)-\Psi(\tfrac{1}{2}-p)\right],\quad p = 1,2,\cdots,$$

and

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

It is not difficult in principle to obtain the coefficients of the higher-order terms, but the labor involved is prohibitive.

---

* On differentiating $P'_{[2\nu_p(\theta_o)-1]/2}(\cos\theta_o) = 0$ implicitly with respect to θ_o we get

$$\frac{d\nu_p(\theta_o)}{d\theta_o} = -\frac{P''_{\nu_p(\theta_o)}(\cos\theta_o)}{\left.\dfrac{\partial P'_{(2\nu-1)/2}(\cos\theta_o)}{\partial\nu}\right|_{\nu=\nu_p(\theta_o)}}.$$

It follows that

$$\left.\frac{d\nu_p(\theta_o)}{d\theta_o}\right|_{\theta_o=\frac{\pi}{2}} = -\frac{P_{[2\nu_p(\pi/2)-1]/2}(0)}{\left(\left.\dfrac{\partial P'_{(2\nu-1)/2}(0)}{\partial\nu}\right|_{\nu=\nu_p(\pi/2)}\right)}.$$

Since $P''_{(2\nu-1)/2}(0) = -(\frac{2\nu-1}{2})(\frac{2\nu+1}{2})P_{(2\nu-1)/2}(0)$ and since explicit expressions for $P_{(2\nu-1)/2}(0)$ and $P_{(2\nu-1)/2}(0)$ as functions of ν are given in [2], Chapter IV, the quantity $\left.\frac{d\nu_p}{d\theta_o}\right|_{\theta_o=\frac{\pi}{2}}$ can be evaluated explicitly [cf.Eq. (5.36)]. The expression for $\left.\frac{d^2\nu_p(\theta_o)}{d\theta_o^2}\right|_{\theta_o=\frac{\pi}{2}}$ is obtained in a similar manner.

6. The Behavior of U and Grad U near the Circular Edge

6.1 Introduction and Summary of Results

We have shown that $U(r,\theta)$ has continuous first and second r- and θ - derivatives and satisfies the potential equation for all points not on the cone. We shall now investigate the behavior of U and ∇U in the neighborhood of the circular edge, $r = 1$, $\theta = \theta_o$, of the cone. The behavior of U is easily disposed of since, from the fact that the integrands in Eqs. (4.26) and (4.26') are $O(\alpha^{-3/2})$ [cf. Appendix II, Eqs. (II-13) and (II-14a)] as $|\alpha|$ approaches infinity on the contour, we can conclude that the corresponding integrals are bounded in the neighborhood of the circular

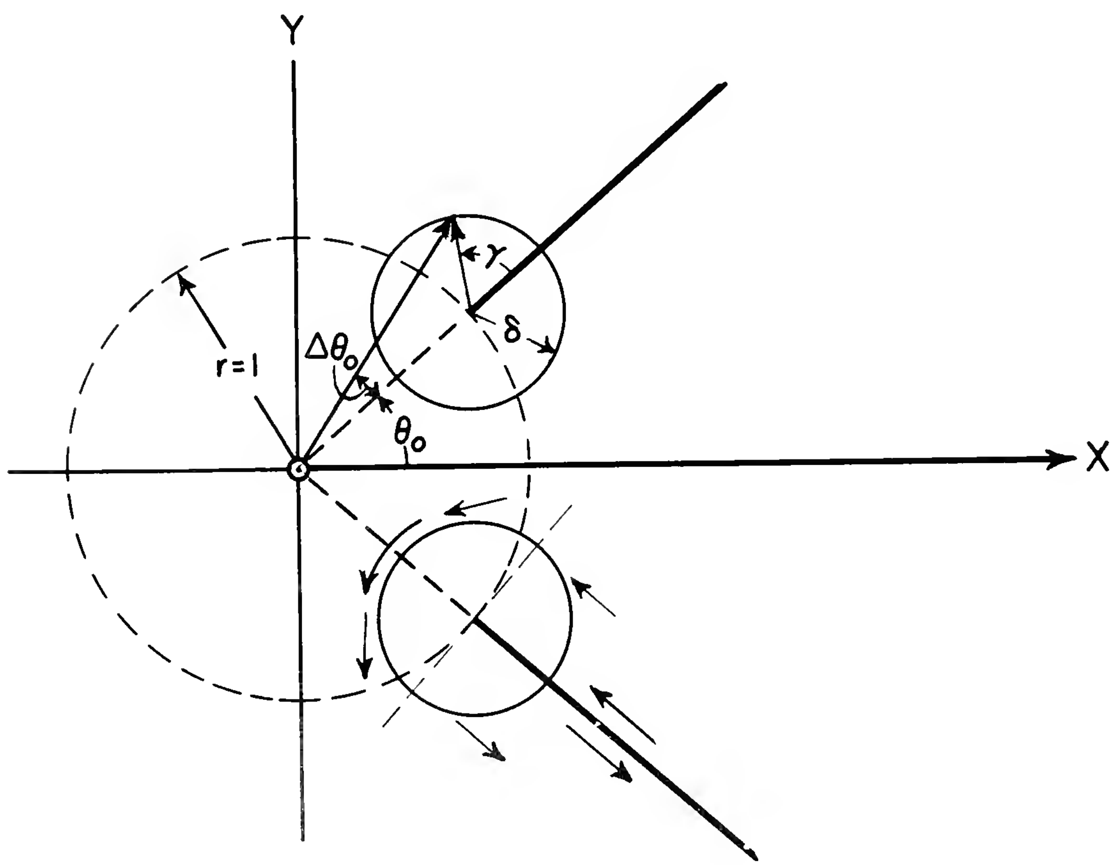

Figure 5

edge and in particular on the edge itself. The main object of the section will therefore be to obtain the behavior of ∇U in the region under consideration. In the remainder of this subsection we give a summary of our results. Then in subsection 6.2 we discuss the method employed to obtain these results.

Let T_δ be a torus whose axis of symmetry is the X-axis [see Fig. 5] and whose equation in the X-Y crossection is

$$(6.1)\qquad T_\delta: \quad \begin{aligned} x &= \cos\theta_0 + \delta\cos(\gamma+\theta_0), \\ y &= \sin\theta_0 + \delta\sin(\gamma+\theta_0), \qquad 0 \leq \gamma \leq 2\pi, \end{aligned}$$

where δ is the radius of the generating circle of the torus. Let $V(x,y) = U(r,\theta)$ and define v_δ and v_γ as follows:

$$(6.2)\qquad \begin{aligned} v_\delta &= \frac{\partial V\left[\cos\theta_0+\delta\cos(\theta_0+\gamma),\ \sin\theta_0+\delta\sin(\theta_0+\gamma)\right]}{\partial\delta} \\ v_\gamma &= \frac{\partial V\left[\cos\theta_0+\delta\cos(\theta_0+\gamma),\ \sin\theta_0+\delta\sin(\theta_0+\gamma)\right]}{\delta\partial\gamma}, \quad 0 \leq \gamma \leq 2\pi. \end{aligned}$$

Then it can be shown that

$$(6.3)\qquad \begin{aligned} \lim_{\delta\to 0}(\delta^{1/2}v_\delta) &\sim \frac{iN(\theta_0)}{\sqrt{2\pi}}\cos\frac{\gamma}{2}, \qquad 0 \leq \gamma \leq 2\pi, \\ \lim_{\delta\to 0}(\delta^{1/2}v_\gamma) &\sim \frac{iN(\theta_0)}{\sqrt{2\pi}}\sin\frac{\gamma}{2}, \qquad 0 \leq \gamma \leq 2\pi, \end{aligned}$$

uniformly in γ for γ in the indicated range. Using the customary notation we may write

$$(6.3')\qquad \begin{aligned} v_\delta &\sim \frac{iN(\theta_0)}{\sqrt{2\pi\delta}}\cos\frac{\gamma}{2}, \qquad 0 \leq \gamma \leq 2\pi, \\ v_\gamma &\sim \frac{iN(\theta_0)}{\sqrt{2\pi\delta}}\sin\frac{\gamma}{2}, \qquad 0 \leq \gamma \leq 2\pi. \end{aligned}$$

It should be recalled that

$$N(\Theta_o) = f/\left(2\pi\sin\Theta_o K_1^+(\Theta_o)\right) \tag{6.4}$$

[cf. Eq. (5.14')], $K_1^+(\Theta_o)$ being defined by Eq. (5.4). As we have already noted, $N(\Theta_o)$ is purely imaginary; the asymptotic expressions for v_δ and v_γ in Eq. (6.3) are therefore real.

When $\Theta_o = \pi/2$, we have from Eq. (5.15)

$$N(\pi/2) = f \,/\, 2i\sqrt{\pi} \tag{6.15}$$

and hence from (6.3)

$$\begin{aligned} v_\delta &\sim \frac{f}{2\pi\sqrt{2\delta}} \cos\frac{\gamma}{2} \quad , \qquad 0 \le \gamma \le 2\pi, \\ v_\gamma &\sim \frac{f}{2\pi\sqrt{2\delta}} \sin\frac{\gamma}{2} \quad , \qquad 0 \le \gamma \le 2\pi. \end{aligned} \tag{6.6}$$

In the following subsection we shall indicate how the above results are obtained. The discussion is limited to obtaining the asymptotic expressions for v_δ and v_γ in the angular range $0 \le \gamma \le \pi$. The corresponding expressions in the angular range $\pi \le \gamma \le 2\pi$ are obtained in a similar manner.

6.2 <u>The Asymptotic Behavior of v_δ and v_Θ as the Circular Edge is Approached in the Region $0 \le \gamma \le \pi$</u>

Writing U_r^2 and U_Θ^2 for $\partial U^2/\partial r$ and $r^{-1}\partial U^2/\partial\Theta$ respectively, we have from Eq. (4.17')

$$\begin{aligned} U_r^2(r,\Theta_o+\Delta\Theta_o) &= -\frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha-1}\,\frac{P_{\alpha-1}[-\cos(\Theta_o+\Delta\Theta_o)]}{P'_{\alpha-1}(-\cos\Theta_o)}\,K^+(\alpha-\tfrac{1}{2},\Theta_o)\,d\alpha, \\ U_\Theta^2(r,\Theta_o+\Delta\Theta_o) &= \frac{N}{2\pi i}\int_{C_\rho} r^{-\alpha-1}\,\frac{P'_{\alpha-1}[-\cos(\Theta_o+\Delta\Theta_o)]}{\alpha P'_{\alpha-1}(-\cos\Theta_o)}\,K^+(\alpha-\tfrac{1}{2},\Theta_o)\,d\alpha, \end{aligned} \tag{6.7}$$

where since $0 \le \gamma \le \pi$ we must have $\pi-\Theta_1 > \Delta\Theta_o \ge 0$. Now from a), p.20 (bottom) it is clear, on substituting $(2\alpha-1)/2$ for ν, that neither integrand in Eq. (6.9)

has a pole at the point $\alpha = 0$. It follows that C_ρ is equivalent to the imaginary α-axis. Let us write

(6.8) $$t = \log r = \log[1 + (r-1)] \quad .$$

From Eq. (6.1) we note that

(6.9) $$r-1 = \sqrt{1 + (2\delta\cos\gamma + \delta^2)} - 1.$$

For small δ we have therefore

(6.10) $$t = \delta(\cos\gamma + \frac{\delta}{2}) + O\left[\delta^2(\cos\gamma + \frac{\delta}{2})\right].$$

If now in Eq. (6.7) we represent $r^{-\alpha-1}$ as $\exp[-(\alpha+1)\log r] = \exp[-\alpha t]\exp(-t)$ and if we make the transformation

$$\alpha = \zeta/\delta,$$

Eq. (6.7) becomes

(6.11)
$$U_r^2(r,\theta_o+\Delta\theta_o) = -\frac{Ne^{-t}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(-t\zeta/\delta)I_r(\zeta/\delta)}{\delta}\,d\zeta,$$

$$U_\theta^2(r,\theta_o+\Delta\theta_o) = \frac{Ne^{-t}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(-t\zeta/\delta)I_\theta(\zeta/\delta)}{\delta}\,d\zeta,$$

where $I_r(\alpha)$ and $I_\theta(\alpha)$ are the r-independent parts of the integrands in the first and second integrals respectively of Eq. (6.7). Now as we let δ approach zero, $\exp(-t\zeta/\delta)$ approaches $\exp(-\cos\gamma\zeta)$ [see Eq. (6.10)], and the argument ζ/δ of I_r and I_θ becomes large in absolute value. It is reasonable to expect that $\exp(-t\zeta/\delta)$ may be replaced by $\exp(-\cos\gamma\zeta)$ and I_r and I_θ by their respective large-argument approximations. The large-argument approximations to I_r and I_θ are the following [see Appendix II, Eqs. (II-12) - (II-14)]:

$$I_r^2(\zeta/\delta) \sim \pm \frac{k^+(\theta_o, \chi_o)\delta^{1/2}}{i\sqrt{\zeta}} \exp\left\{\pm i \left[\Delta\theta_o \frac{\zeta}{\delta} + \frac{\Delta\theta_o}{2}\right]\right\},$$

(6.12)

$$I_\theta^2(\zeta/\delta) \sim \frac{k^+(\theta_o, \chi_o)\delta^{1/2}}{\sqrt{\zeta}} \exp\left\{\pm i \left[\Delta\theta_o \frac{\zeta}{\delta} + \frac{\Delta\theta_o}{2}\right]\right\},$$

where the upper sign is employed when $\mathrm{Im}(\zeta) \geq 0$ and the lower sign when $\mathrm{Im}(\zeta) \leq 0$, and where

(6.12') $$\sqrt{\zeta} = |\zeta| \exp[i \arg(\zeta/2)], \qquad -\pi \leq \arg \zeta \leq \pi.$$

From an inspection of Fig. 5 it is clear that

(6.13) $$\Delta\theta_o = \delta \sin\gamma + O(\delta^2), \qquad 0 \leq \gamma \leq 2\pi \quad .$$

Assuming it is permissible to replace $\exp\left[\pm i(\zeta\, O(\delta^2) + \frac{\Delta\theta_o}{2})\right]$ in Eq. (6.12) by unity we have

$$I_r^2(\zeta/\delta) \sim \pm \frac{k^+(\theta, \chi_o)\delta^{1/2}}{i} \frac{\exp(\pm i \zeta \sin\gamma)}{\sqrt{\zeta}}, \quad 0 \leq \gamma \leq \pi,$$

(6.14)

$$I_\theta^2(\zeta/\delta) \sim k^+(\theta, \chi_o)\delta^{1/2} \exp(\pm i \zeta \sin\gamma), \qquad 0 \leq \gamma \leq \pi.$$

If we substitute these results into Eq. (6.11) we get for $0 \leq \gamma \leq \pi$

$$U_r^2 \sim -\frac{Nk^+(\theta_o, \chi_o)}{i(2\pi i)\delta^{1/2}} \left[\int_0^{i\infty} \frac{\exp[-\zeta e^{-i\gamma}]d\zeta}{\sqrt{\zeta}} + \int_0^{-i\infty} \frac{\exp[-\zeta e^{i\gamma}]d\zeta}{\sqrt{\zeta}} d\zeta\right],$$

(6.15)

$$U_\theta^2 \sim \frac{Nk^+(\theta_o, \chi_o)}{(2\pi i)\delta^{1/2}} \left[\int_0^{i\infty} \frac{\exp[-\zeta e^{-i\gamma}]d\zeta}{\sqrt{\zeta}} - \int_0^{-i\infty} \frac{\exp[-\zeta e^{i\gamma}]d\zeta}{\sqrt{\zeta}} d\zeta\right].$$

The derivation of Eq. (6.15) from Eq. (6.11) will be justified rigorously in Appendix IV where we prove that

$$\lim_{\delta\to 0}(\delta^{\frac{1}{2}}U_r^2) = -\frac{Nk^+(\theta_o,\chi_o)}{2\pi i^2}\left[\int_0^{i\infty}\frac{\exp[-\zeta e^{-i\gamma}]}{\sqrt{\zeta}}d\zeta + \int_0^{-i\infty}\frac{\exp[-\zeta e^{i\gamma}]}{\sqrt{\zeta}}d\zeta\right],$$

(6.15')

$$\lim_{\delta\to 0}(\delta^{\frac{1}{2}}U_\theta^2) = \frac{Nk^+(\theta_o,\chi_o)}{2\pi i}\left[\int_0^{i\infty}\frac{\exp[-\zeta e^{-i\gamma}]}{\sqrt{\zeta}}d\zeta - \int_0^{-i\infty}\frac{\exp[-\zeta e^{i\gamma}]}{\sqrt{\zeta}}d\zeta\right]$$

uniformly in γ for γ on the range $0 \le \gamma \le \pi$.

Now let

$$P = \int_0^{i\infty}\frac{\exp(-\zeta e^{-i\gamma})}{\sqrt{\zeta}}d\zeta.$$

It is easy to show that $\mathrm{Re}(-\zeta e^{-i\gamma}) \le 0$ when $\arg\zeta$ is in the range $\frac{\pi}{2} + \gamma \le \arg\zeta = -\frac{\pi}{2} + \gamma$. Let us deform the contour to the ray $\arg\zeta = \gamma$. Since no poles or branch cuts are encountered in this deformation we have

$$P = \int_0^{\infty, e^{i\gamma}}\frac{\exp(-\zeta e^{-i\gamma})}{\sqrt{\zeta}}d\zeta = \exp(\tfrac{i\gamma}{2})\int_0^\infty\frac{e^{-|\zeta|}}{\sqrt{|\zeta|}}d\zeta = \sqrt{\pi}\exp(\tfrac{i\gamma}{2}).$$

In a similar manner it may be shown that

$$\int_0^{-i\infty}\frac{\exp(-\zeta e^{+i\gamma})}{\sqrt{\zeta}}d\zeta = \sqrt{\pi}\ \exp(\tfrac{-i\gamma}{2}).$$

Thus we have

$$U_r^2 \sim +\frac{Nk^+(\theta_o,\chi_o)}{\sqrt{\pi\delta}}\cos(\gamma/2), \qquad 0 \le \gamma \le \pi,$$

(6.16)

$$U_\theta^2 \sim +\frac{Nk^+(\theta_o,\chi_o)}{\sqrt{\pi\delta}}\sin(\gamma/2), \qquad 0 \le \gamma \le \pi,$$

which, since

(6.17) $\qquad k^+(\theta_o,\chi_o) = i/\sqrt{2}$,

[see Appendix II, Eq. (II-21)], becomes

$$U_r^2 \sim \frac{iN}{\sqrt{2\pi\delta}} \cos\frac{\gamma}{2}, \qquad 0 \le \gamma \le \pi$$

(6.18)

$$U_\theta^2 \sim \frac{iN}{\sqrt{2\pi\delta}} \sin\frac{\gamma}{2}, \qquad 0 \le \gamma \le \pi.$$

Expressing $\partial V^2/\partial x$ and $\partial V^2/\partial y$ in terms of U_r^2 and U_θ^2, where $V^2(x,y) = U^2(r,\theta)$, and then expressing $\partial V^2/\partial\delta = v_\delta$ and $\delta^{-1}(\partial V^2/\partial\gamma) = v_\gamma$ in terms of $\partial V^2/\partial x$ and $\partial V^2/\partial y$, it is easy to verify that

$$v_\delta \sim \frac{iN(\theta_o)}{\sqrt{2\pi\delta}} \cos(\gamma/2), \qquad 0 \le \gamma \le \pi,$$

(6.19)

$$v_\gamma \sim \frac{iN(\theta_o)}{\sqrt{2\pi\delta}} \sin(\gamma/2), \qquad 0 \le \gamma \le \pi.$$

These are the formulas quoted in Eq. (6.5) for the γ-interval under consideration. As already noted the above formulas are also valid in the interval $\pi \le \gamma \le 2\pi$.

7. Uniqueness of the Solution

In the previous section we showed that the quantities U and $\frac{\partial U}{\partial\delta} = v_\delta$ are of $O(1)$ and $O(\delta^{-1/2})$ respectively, in the neighborhood of the circular edge of the cone. Since these estimates hold uniformly for γ, $0 \le \gamma \le 2\pi$ it follows that

(7.1) $$\lim_{\delta\to 0} \int_{sT_\delta} U \frac{\partial U}{\partial\delta}\, ds = \lim_{\delta\to 0} \int_{sT_\delta} Uv_\delta\, ds = 0,$$

where sT_δ stands for the surface of the torus T_δ defined by Eq. (6.1). Let V be any velocity potential which satisfies Eq. (7.1) and conditions (2.4a)-

(2.4c), and which has the following behavior (a) 'inside' and (b) 'outside' the conical pipe:

$$(a) \quad \frac{\partial V}{\partial r} \sim - \frac{f}{4\pi \sin^2\theta_o r^2}, \qquad r \to \infty, \qquad 0 \leq \theta \leq \theta_o$$

(7.2)

$$(b) \quad \frac{\partial V}{\partial r} \sim \frac{f}{4\pi \cos^2\theta_o r^2}, \qquad r \to \infty, \qquad \theta_o \leq \theta \leq \pi.$$

Thus both U and V are solutions of Laplace's equation and satisfy the appropriate regularity conditions, and in addition are characterized by the same flux conditions at infinity. If we write W = U - V, then clearly

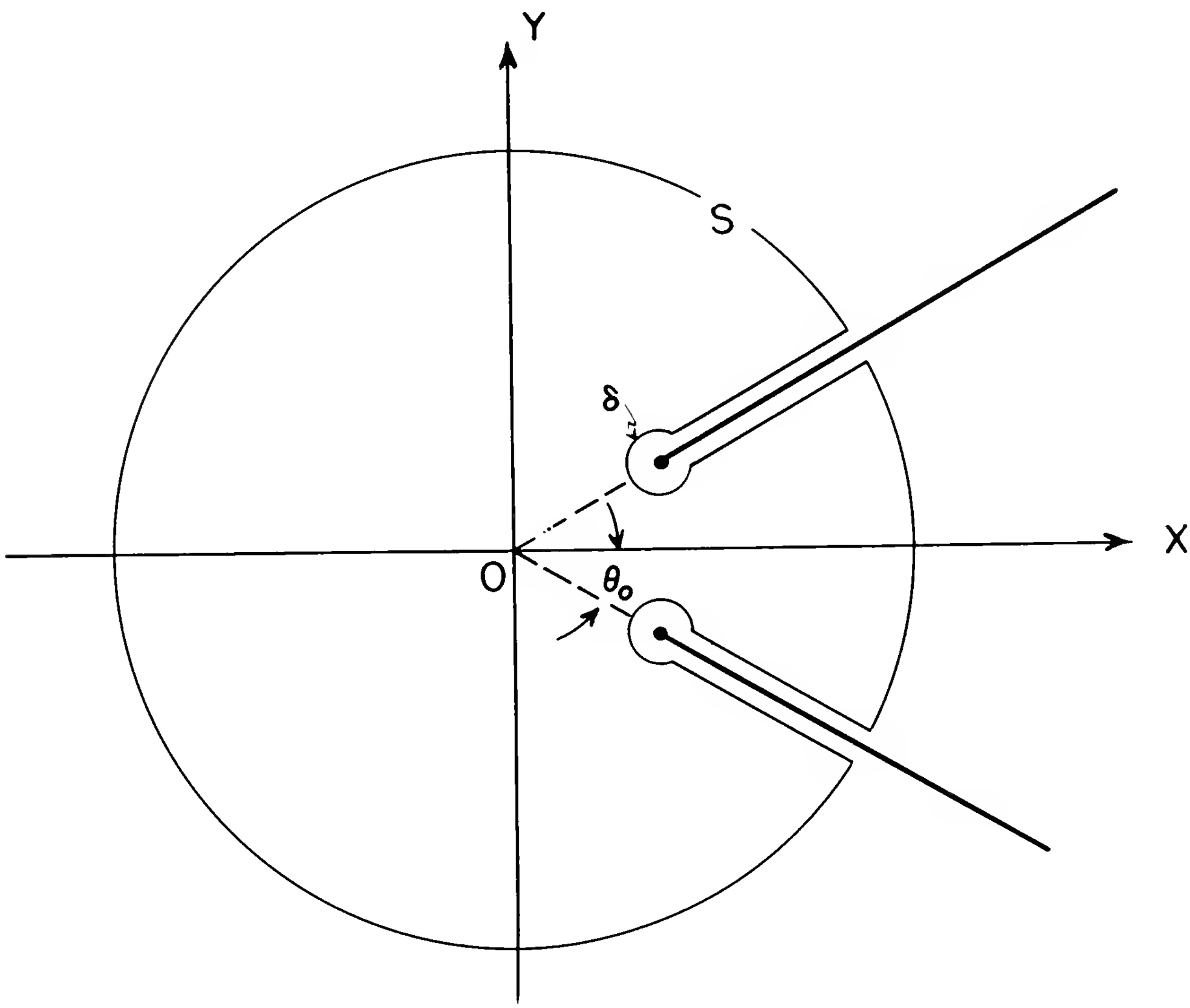

Figure 6

(7.3) $$\int_{vS} W\,\nabla^2 W\,dv = \int_{sS} W\,\nabla W \cdot d\bar{s} - \int_{vS} (\nabla W)^2 dv,$$

where vS and sS stand for the volume and surface areas respectively of the large indented spherical surface S centered at the origin. A cross-sectional view of S is shown in Fig. 6. Now in view of the properties of the velocity potentials U and V we see that if we let the radius of the sphere S become infinite and let δ approach zero we have

(7.4) $$\int (\nabla W)^2\,dv = 0,$$

the integral being taken over all space. It follows that

(7.5) $$U = V + \text{constant}.$$

Thus, except for an arbitrary and active constant, U is the only function satisfying conditions (2.4a) - (2.4c), Eq. (7.1) and the flux conditions of Eq. (7.2).

At the end of subsection 5.1 we observed that the incoming flux was the same as the outgoing flux. It is interesting to note that this fact implies that flux is neither emitted nor absorbed by the circular edge. The converse is also true. To prove these assertions let

(7.6) $$\begin{aligned} \frac{\partial U}{\partial r} &\sim -\frac{f}{4\pi \sin^2\theta_o}\,\frac{1}{r^2}, \qquad r \to \infty, \quad 0 \leq \theta \leq \theta_o, \\ \frac{\partial U}{\partial r} &\sim \frac{f'}{4\pi \cos^2\theta_o}\,\frac{1}{r^2}, \qquad r \to \infty, \quad \theta_o \leq \theta \leq \pi. \end{aligned}$$

Then from the relation

(7.7) $$0 = \int_{vS} \nabla^2 U = \int_{sS} \frac{\partial U}{\partial n}\,ds$$

and the fact that the normal derivative $\partial U/\partial n$ vanishes on the surface of the

cone, we have on letting the radius of S become infinite and δ approach zero,

$$f' - f = \lim_{\delta \to 0} \int_{sT_\delta} \frac{\partial U}{\partial \delta}\, ds \,. \tag{7.8}$$

Thus $f' = f$ is a necessary and sufficient condition that flux is neither emitted nor absorbed by the edge of the cone.

8. Application of the Rayleigh Static Method

In this section we employ the Rayleigh procedure to obtain the far field outside ($\theta_0 \leq \theta \leq \pi$) the cone which arises from a given exciting field of sound waves inside ($0 \leq \theta \leq \theta_0$) the cone. The main feature of the method consists in recognizing that the field in the neighborhood of the aperture is essentially the same as that of the potential-flow problem treated in the previous sections when the wave-length is small in comparison to the dimensions of the aperture. It will be seen that under these circumstances, only the lowest exciting mode contributes to the far field outside the cone and that the far field depends only on a gross characteristic of the static field, namely the conductivity of the aperture. The account of Rayleighs' procedure which follows will be purely formal in nature; we make no claims to rigor.

Let U be an axially symmetric velocity potential which has a vanishing normal derivative on the cone and which satisfies the time-independent wave equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\,\frac{\partial U(r,\theta)}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\,\frac{\partial U(r,\theta)}{\partial \theta}\right] + k^2 U = 0, \tag{8.1}$$

k being the wave number. Let U^1 $[U^2]$ denote U in the conical region T^1 $[T^2]$

defined by $0 \leq \Theta \leq \Theta_o$ $[\Theta_o \leq \Theta \leq \pi]$; in addition let $G_k^1(\bar{r},\bar{r}')$ $[G_k^2(\bar{r},\bar{r}')]$ be the Green's function for T^1 $[T^2]$ that has a vanishing normal derivative on the cone and satisfies the radiation condition at infinity, and finally let C_b be the surface of the conical cup defined by $\Theta = \Theta_o$, $0 \leq r \leq b$. If U arises from excitation inside the cone of the form

$$(8.1') \qquad U_o^1 = \sum_{n=1}^{N} g_n \sqrt{\frac{\pi}{2kr}}\, H^{(1)}_{(2\mu_n+1)/2}(kr) P_{\mu_n}(\cos\Theta), \qquad 0 \leq \Theta \leq \Theta_o,$$

then U^1 and U^2 can be represented as follows:

$$(8.2) \quad U^1 = 2\sum_{n=1}^{N} g_n \sqrt{\frac{\pi}{2kr}}\, J_{(2\mu_n+1)/2}(kr) P_{\mu_n}(\cos\Theta) - \int_{C_b} G_k^1(\bar{r},\bar{r}') \frac{\partial U^1}{\partial\nu}(\bar{r}')ds(\bar{r})$$

$$(8.2') \quad U^2 = + \int_{C_b} G_k^2(\bar{r},\bar{r}') \frac{\partial U^2(r')}{\partial\nu}\, ds(\bar{r}').$$

The terms $\sqrt{\pi/2kr}\, J_{(2\mu_n+1)/2}(kr)P_{\mu_n}(\cos\Theta)$, $\sqrt{\pi/2kr}\, H^{(1)}_{(2\mu_n+1)/2}(kr)\, P_{\mu_n}(\cos\Theta)$, $n = 1,\cdots,N$, involving the Bessel and Hankel functions of order $(2\mu_n+1)/2$ and the Legendre functions of degree μ_n are product solutions of Eq. (8.1) obtained by means of the usual separation of variable procedure. The quantities μ_n, are the roots of the equation $\left.\frac{dP_\mu(\cos\Theta)}{d\Theta}\right|_{\Theta=\Theta_o} = 0$. $\frac{\partial U^1}{\partial\nu}$ and $\frac{\partial U^2}{\partial\nu}$ are the derivatives of U^1 and U^2 in the direction of the unit normal $\bar{\nu}$ to the surface of C_b; it is assumed that $\bar{\nu}$ points into T^1. In the following we shall assume that the g_n are constants independent of k, that $g_1 \neq 0$ and $\mu_1 = \mu_1(\Theta_o) = 0$.*

---

* Note that $P_o(\cos\Theta) = 1$; hence

$$\frac{dP_o(\cos\Theta)}{d\Theta} = 0 .$$

The term involving the summation in the right hand side of Eq. (8.2) may be written in the following form:

$$\sum_{n=1}^{N} g_n \sqrt{\frac{\pi}{2kr}}\, H^{(1)}_{(2\mu_n+1)/2}(kr) P_{\mu_n}(\cos\theta) + \sum_{n=1}^{N} g_n H^{(2)}_{(2\mu_n+1)/2}(kr) P_{\mu_n}(\cos\theta).$$

Assuming the harmonic time dependence $e^{+i\omega t}$ then in virtue of the asymptotic behavior of the Hankel functions we may interpret the first sum as an incident field and the second sum as that reflected field which would be present were there no aperture in the cone. The remaining term in (8.2) represents outgoing waves in T^1 contributed by the aperture to the total field. In T^2 [see Eq. (8.2')], the aperture contribution is the sole contribution to the field. It should be added that N may be infinite provided the g_n are such that the infinite sum has the proper convergence properties. In particular, in the case where the cone reduces to a plane screen with a circular aperture, the sum term in Eq. (8.2) may be regarded as the eigenfunction expansion of the plane incident and reflected waves appearing in the sum $e^{ikx} + e^{-ikx}$.

We now turn to the problem of obtaining the far field in the region T^2 under the assumption that the wavelength is large compared to the dimensions of the conical cup C_b. Under these circumstances the Green's function $G^2_k(\bar{r},\bar{r}')$ appearing in Eq. (8.2') reduces to*

*Employing the standard method for obtaining Green's functions in coordinate systems which allow separation of variables, we have

$$G^2_k(r,\theta,\phi;r',\theta',\phi') = -\frac{ik}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{P^{|m|}_{\mu_n}(-\cos\theta)e^{im(\phi-\phi')}}{\int_{\theta_0}^{\pi} P^{|m|}_{\mu_n}(-\cos\theta)^2 \sin\theta\, d\theta}\,\cdot$$

$$\begin{Bmatrix} \sqrt{\frac{\pi}{2kr'}}\, H^{(2)}_{(2\mu_n+1)/2}(kr')\sqrt{\frac{\pi}{2kr}}\; J_{(2\mu_n+1)/2}(kr) \\[2ex] \sqrt{\frac{\pi}{2kr}}\; J_{(2\mu_n+1)/2}(kr')\sqrt{\frac{\pi}{2kr}}\; H^{(2)}_{(2\mu_n+1)/2}(kr) \end{Bmatrix}$$

where the upper line is employed when $r < r'$ and the lower line when $r > r'$.

(continued on bottom of next page)

$$G_k^2(r,\theta,\phi;r',\theta',\phi') \sim + \frac{1}{2\pi(1+\cos\theta_o)} \frac{e^{-ikr}}{r}, \quad kr \to \infty. \tag{8.3}$$

The far field in region T^2 is therefore

$$U^2 \sim + \frac{1}{2\pi(1+\cos\theta_o)} \frac{e^{-ikr}}{r} \int_{C_b} \frac{\partial U^2(\bar{r}')}{\partial \nu} \, ds(\bar{r}'). \tag{8.4}$$

Thus the problem of finding the far field of $U^2(r)$ is reduced to the problem of finding the integral over the surface C_b of the velocity component of the fluid normal to it. Now since U is by hypothesis a solution to the problem, $U = U^1 = U^2$ and $\frac{\partial U}{\partial \nu} = \frac{\partial U^1}{\partial \nu} = \frac{\partial U^2}{\partial \nu}$ on C_b. These relations yield, in virtue of Eqs. (8.2) and (8.2') the following integral equation:

$$0 = 2\sum_{n=1}^{\infty} g_n \sqrt{\frac{\pi}{2kr}}\, J_{(2\mu_n+1)/2}(kr) P_{\mu_n}(\cos\theta) + \int_{C_b} \left[G_k^2(\bar{r},\bar{r}') + G_k^1(\bar{r},\bar{r}')\right] \frac{\partial U(\bar{r}')}{\partial \nu} \, ds(\bar{r}'), \tag{8.5}$$

where both $\bar{r}$ and $\bar{r}'$ are on C_b. When the wavelength is large compared to the dimensions of C_b this integral equation reduces, in virtue of the small-argument behavior of Bessel functions, to

$$0 = 2g_1 - \int_{C_b} \left[G_o^1(\bar{r},\bar{r}') + G_o^2(\bar{r},\bar{r}')\right] \frac{\partial U(\bar{r}')}{\partial \nu} \, ds(\bar{r}'), \tag{8.6}$$

where $G_o^1(\bar{r},\bar{r}')$ and $G_o^2(\bar{r},\bar{r}')$* are the static Green's functions inside and outside the cone, respectively.

(continued from previous page)
The $P^m_{\mu_n}(-\cos\theta)$ are the associated Legendre functions which satisfy $\frac{dP^m_{\mu_n}(-\cos\theta)}{d\theta} = 0$ when $\theta = \theta_o$. Eq. (8.3) is obtained from the lower line by making use of the well-known behavior of Bessel functions for small arguments and Hankel functions for large arguments.

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* G_o^2 is obtained formally from the expression for $G_k^2(\bar{r},\bar{r}')$ given in the previous footnote by employing the small argument formulas for the Hankel and Bessel functions involved there. $G_o^1(\bar{r},\bar{r}')$ is obtained from the corresponding expression for $G_k^1(\bar{r},\bar{r}')$ in a similar manner.

As we have mentioned above, the main feature of the Rayleigh procedure consists in recognizing that the integral equation (8.6) is the same as that which would be obtained from the problem we have treated in Section 1 to 6. That this is indeed the case can be shown as follows: Let U satisfy Laplace's equation (2.3) and the conditions of Eq. (2.4), and as $r \to \infty$ let U approach the constant potentials U_o^1 and $U_o^2 \neq U_o^1$ in the regions T^1 and T^2 respectively. It is easily seen that if we now employ the static Green's functions $G_o^2(\bar{r},\bar{r}')$ and $G_o^1(\bar{r},\bar{r}')$ mentioned above and proceed by the method given in this section we are led to the integral equation

$$(8.7) \qquad 0 = U_o^1 - U_o^2 - \int_{C_b} \left[G_o^1(\bar{r},\bar{r}') + G_o^2(\bar{r},\bar{r}')\right] \frac{\partial U(\bar{r}')}{\partial \nu} \, ds(\bar{r}'),$$

where both $\bar{r}$ and $\bar{r}'$ are on C_b. Thus $\frac{\partial U}{\partial \nu}$ in Eq. (8.6) may be interpreted as the velocity normal to C_b of an incompressible fluid undergoing steady irrotational flow, provided we set

$$(8.8) \qquad 2g_1 = U_o^1 - U_o^2 .$$

We shall ultimately compare our results with those of Rayleigh, as presented by Lamb in [5], p. 518, for the special case when the cone becomes a plane screen. It is therefore necessary to note at this point that Lamb employs $-\nabla U$ for the velocity of the fluid whereas, in the preceding sections, we have employed ∇U for this quantity. It is, however, easily verified, on substituting -f for f, that the resulting U is a solution of the potential problem arising from incoming flux at infinity in the region T^1 [cf. Eqs. (2.4d) and (2.5)], $-\nabla U$ being the velocity of the fluid. In this case, if we define the conductivity $\sigma(\theta_o)$ as $f/(U_o^1 - U_o^2)$ it can be shown, employing the reasoning of Section 5.3, that $U_o^1 > U_o^2$, i.e., that the fluid flows from the higher to the lower potential, that $\sigma(\theta_o)$ is a positive quantity and that the expression for $\sigma(\theta_o)$ is identical with the one we have given in Section 5.3 Eq. (5.31). In the following we shall employ Lamb's convention with regard to the velocity potential and furthermore assume that $g_1 > 0$.

Now since the flux passing into region T^2 through C_b is the same as the incoming flux f at infinity in T^1 we have for a fluid of unit density

(8.9) $$f \sim \int_{C_b} \frac{\partial U^2}{\partial \nu} \, ds(\bar{r}')$$

where, it should be recalled, $\frac{\partial U^2}{\partial \nu}$ is the derivative in the direction of the unit normal, $\bar{\nu}$, to C_b which points into T^1. From the definition of the conductivity $\sigma(\theta_o)$ given above and from Eqs. (8.8) we therefore have

(8.10) $$f = 2g_1 \, \sigma(\theta_o),$$

where an explicit expression for $\sigma(\theta_o)$ is given in Eq. (5.31). Thus, as r approaches infinity in T^2, we have from Eq. (8.4)

(8.11) $$U^2 \sim \frac{g_1 \sigma(\theta_o)}{(1+\cos\theta_o)} \; \frac{e^{-ikr}}{r} .$$

In the case where the cone reduces to a plane screen with a circular aperture we have, on setting $\theta_o = \pi/2$ in Eq. (8.10),

(8.12) $$g_1 = 1, \quad \sigma(\pi/2) = 2b$$

[see Eq. (5.33) and the corresponding footnote]. Hence it follows that

(8.13) $$U^2 \sim \frac{2b}{\pi} \; \frac{e^{-ikr}}{r} , \; x < 0.$$

In region T^1 it can be shown using Eqs. (8.2) and (8.2') that

(8.14) $$U^1 \sim e^{ikx} + e^{-ikx} - \frac{2b}{\pi} \; \frac{e^{-ikr}}{r}, \; x > 0.$$

These results agree with those of Rayleigh. [See [5], p. 518, Eqs. (10), (17), (18) and (20)].

Appendix I

The Asymptotic Behavior of $\prod(\nu,\theta_o)$ in the Region $|\arg(\nu + \frac{1}{2})| < \pi$

We shall prove the following: As $|\nu| \to \infty$ in the region $|\arg(\nu + \frac{1}{2})| < \pi$,

$$\prod(\nu,\theta_o) \sim \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{\theta_o \nu}{\pi} + \frac{1}{4})} L(\theta_o) \exp\left\{\frac{\theta_o \nu}{\pi}\left(M(\theta_o) + \Psi(\tfrac{1}{4})\right)\right\}, \tag{I-1}$$

where

$$\left\{\begin{aligned} \Psi(z) &= \Gamma'(z)/\Gamma(z) \\ L(\theta_o) &= \prod_{p=1}^{\infty}\left\{\frac{(p - \frac{3}{4})}{\theta_o \cdot \nu_p(\theta_o)/\pi}\right\} \\ M(\theta_o) &= \prod_{p=1}^{\infty}\left(\frac{1}{p - \frac{3}{4}} - \frac{1}{\theta_o \cdot \nu_p(\theta_o)/\pi}\right) . \end{aligned}\right. \tag{I-2}$$

To prove this statement we compare the growth of $\prod(\nu,\theta_o)$ with that of the function

$$Q\left(\frac{\theta_o \nu}{\pi}\right) = \prod_{p=1}^{\infty}\left\{\left(1 + \frac{\frac{\theta_o \nu}{\pi}}{(p - \frac{3}{4})}\right) \exp\left[-\left(\frac{\theta_o \nu}{\pi}\Big/(p - \tfrac{3}{4})\right)\right]\right\} . \tag{I-3}$$

This function, in turn, can be related to the gamma function by the formula

$$\frac{\Gamma(a+1)}{\Gamma(z+a+1)} \cdot e^{z\Psi(a+1)} = \prod_{p=1}^{\infty}\left\{\left(1 + \frac{z}{p+a}\right) \exp(-z/p+a)\right\} . \tag{I-4}$$

Clearly

$$Q(\nu) = \frac{\Gamma(\frac{1}{4})}{\Gamma(\nu + \frac{1}{4})} \exp(\nu\, \Psi(\tfrac{1}{4})) , \tag{I-5}$$

from which it follows that

$$\text{(I-6)} \qquad \prod(\nu,\Theta_o)/Q(\frac{\Theta_o\nu}{\pi}) = \frac{\prod_{p=1}^{\infty}\left\{\left(1+\frac{\nu}{\nu_p(\Theta_o)}\right)\exp(\frac{-\nu}{\nu_p(\Theta_o)})\right\}}{\prod_{p=1}^{\infty}\left\{\left(1+\frac{\nu}{\frac{\pi}{\Theta_o}(p-\frac{3}{4})}\right)\exp\left(-\frac{\nu}{\frac{\pi}{\Theta_o}(p-\frac{3}{4})}\right)\right\}}$$

Calling the left-hand side of (I-6) $R(\nu)$ and taking the logarithm of both sides we get

$$\text{(I-7)} \qquad \log R(\nu) = \sum_{p=1}^{\infty}\log\left\{\left(\frac{1+\frac{\nu}{\nu_p(\Theta_o)}}{1+\frac{\nu}{\frac{\pi}{\Theta_o}(p-\frac{1}{4})}}\right)\right\} + \sum_{p=1}^{\infty}\left(\frac{\nu}{\frac{\pi}{\Theta_o}(p-\frac{3}{4})} - \frac{\nu}{\nu_p(\Theta_o)}\right),$$

since each infinite series in (I-7) converges separately for all ν as a consequence of the arithmetic growth of the roots $\nu_p(\Theta_o)$ [see Eq. (3.12a) or Eq. (I-11)]. It follows that

$$\text{(I-8)} \qquad R(\nu) = \prod_{p=1}^{\infty}\left\{\frac{1+\frac{\nu}{\nu_p(\Theta_o)}}{1+\frac{\nu}{\frac{\pi}{\Theta_o}(p-\frac{3}{4})}}\right\} \cdot \exp\left[\frac{\Theta_o}{\pi}\nu M(\Theta_o)\right],$$

where $M(\Theta_o)$ is defined in Eq. (I-2).

We need only show now that the first factor of the right-hand side of Eq. (I-8) approaches the constant $L(\Theta_o)$ as $|\nu| \to \infty$ in the region $|\nu + \frac{1}{2}| < \pi$. Let us denote this factor by $N(\nu)$ and make the substitution

$$\text{(I-9)} \qquad \nu + \frac{1}{2} = \frac{1}{\mu} .$$

Thus we have

$$\text{(I-10)} \qquad N(\frac{1}{\mu}-\frac{1}{2}) = \prod_{p=1}^{\infty}\left\{\frac{\frac{\pi}{\Theta_o}(p-\frac{3}{4})}{\nu_p(\Theta_o)}\right\}\left\{\frac{(2\mu)\nu_p(\Theta_o)+(2-\mu)}{2\mu(p-\frac{3}{4})\frac{\pi}{\Theta_o}+(2-\mu)}\right\} .$$

Now for large p we have from Eq. (3.12a)

$$\nu_p(\Theta_o) \sim \frac{\pi}{\Theta_o}\left(p - \frac{3}{4}\right) + \frac{C(\Theta_o)}{p} \quad . \tag{I-11}$$

It is then an easy matter to prove the convergence of the infinite product

$$L(\Theta_o) = \prod_{p=1}^{\infty} \left\{ \frac{\frac{\pi}{\Theta_o}(p - \frac{3}{4})}{\nu_p(\Theta_o)} \right\} \quad . \tag{I-12}$$

Furthermore, calling the infinite product of the last two factors in Eq. (I-10) $S(\mu)$, we can prove that $S(\mu)$ converges uniformly for all μ in the neighborhood of $\mu = 0$ and $|\arg \mu| \leq \pi + \varepsilon$. Let us rewrite $S(\mu)$ in the following form

$$S(\mu) = \prod_{p=1}^{\infty} \left\{ 1 + \frac{2\mu\left[-\frac{\pi}{\Theta_o}(p - \frac{3}{4}) + \nu_p(\Theta_o)\right]}{2\mu(p - \frac{3}{4})\frac{\pi}{\Theta_o} + (2 - \mu)} \right\} \quad . \tag{I-13}$$

Now define $f_p(\mu)$ by the equation

$$f_p(\mu) = \frac{2\mu\left[\nu_p(\Theta_o) - \frac{\pi}{\Theta_o}(p - \frac{3}{4}\right]}{2\mu(p - \frac{3}{4})\frac{\pi}{\Theta_o} + (2-\mu)} \qquad p = 1,2,\ldots \quad . \tag{I-14}$$

Then $S(\mu)$ converges uniformly in the neighborhood of $\mu = 0$ and $|\arg \mu| < \pi + \varepsilon$ if and only if the series

$$\sum_{p=1}^{\infty} |f_p(\mu)| \tag{I-15}$$

converges uniformly for μ in this region.

Now for large p we have as a consequence of Eq. (I-11)

$$|f_p(\mu)| \quad \frac{p^{-1}|C(\Theta_o)|}{|\left[(p - \frac{3}{4})(\pi/\Theta_o) - 2^{-1} + \mu^{-1}\right]|} \quad . \tag{I-16}$$

We shall call the denominator of the last expression D; it is the distance of the point $z = \mu^{-1}$ from the point $z = -(p - \frac{3}{4})(\pi/\Theta_o) + (\frac{1}{2})$. Let ε be any positive number, arbitrarily small, and suppose

(I-17) $$-\pi + \varepsilon \leq \arg \mu \leq \pi - \varepsilon .$$

Clearly the minimum value of D is $\left[(p - \frac{3}{4}) \frac{\pi}{\Theta_o} - (\frac{1}{2})\right] \sin \varepsilon$. It follows that for sufficiently large p

(I-18) $$\left|f_p(\mu)\right| \leq \frac{C(\Theta_o)}{p\left[(p - \frac{3}{4}) \frac{\pi}{\Theta_o} - (\frac{1}{2})\right] \sin \varepsilon} , -\pi + \varepsilon \leq \arg \mu \leq \pi - \varepsilon .$$

As a result the infinite product expansion of $S(\mu)$ given in Eq. (I-13) converges in the neighborhood of $\mu = 0$ for $-\pi + \varepsilon \leq \arg \mu \leq \pi - \varepsilon$. Consequently, as μ approaches zero in this region,

(I-19) $$\lim_{\mu \to 0} S(\mu) = 1 .$$

It follows from Eqs. (I-10) and (I-12) that $\lim N(\mu^{-1} - 2^{-1}) = L(\Theta_o)$, $-\pi + \varepsilon \leq \arg \mu \leq \pi - \varepsilon$, and thus we have, employing Eq. (I-9),

(I-20) $$\lim_{\nu \to \infty} N(\nu) = L(\Theta_o), \qquad -\pi + \varepsilon \leq \arg(\nu + \tfrac{1}{2}) \leq \pi - \varepsilon .$$

From Eq. (I-6) we have therefore

(I-21) $$\Pi(\nu,\Theta_o) \sim Q(\frac{\Theta_o}{\pi} \nu)L(\Theta_o) \exp\left[\frac{\Theta_o}{\pi} \nu M(\Theta_o)\right]$$

as $|\nu| \to \infty$ in the region $-\pi + \varepsilon \leq \arg(\nu + \frac{1}{2}) \leq \pi - \varepsilon$. Employing Eq. (I-5) we find that the right-hand side of Eq. (I-21) reduces to the right-hand side of Eq. (I-1). The validity of the latter equation in the indicated region has therefore been demonstrated.

Appendix II

On the Convergence of the Integral Representations for the Velocity Potential and its Various Derivatives.

The object of this appendix is to verify the statements (a) through (g), pp. (22) to (26) and to obtain the asymptotic expressions required in Sec. 6. To this end we shall first discuss certain asymptotic expressions which will recur frequently in our subsequent calculations.

We shall assume throughout the following that the $\beta = \arg\alpha$ and that the polar form of α is

$$\text{(II-1)} \qquad \alpha = |\alpha|\, e^{i\beta} \qquad -\pi < \beta \leq \pi \, .$$

Now from [2] p. 71

$$\text{(II-2)} \qquad P^{\mu}_{\alpha-1}(\cos\theta) = (\alpha - 1)^{\mu} \sqrt{\frac{2}{\pi(\alpha-1)\sin\theta}}\, \cos\left[(\alpha - \tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{\mu\pi}{2}\right] + O\left[(\alpha-1)^{-3/2}\right]$$

for real values of μ and for $|\alpha-1| \gg 1$, $|\alpha-1| \gg |\mu|$ and $\arg|\alpha-1| < \pi$, $\left[\varepsilon \leq \theta \leq \pi - \varepsilon,\ \varepsilon > 0,\ |\alpha - 1| \gg \frac{1}{\varepsilon}\right].$ The α plane is cut from $\alpha = -\infty$ to $\alpha = +1$ and the square root in Eq. (II-2) is taken positive whenever $\alpha/\pi(\alpha - 1)\sin\theta_0$ is positive.

Now for α not real, that is for $\beta \neq 0$ or π

$$\text{(II-3)} \qquad \cos\left[(\alpha - \tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{\mu\pi}{4}\right] \sim 2^{-1} \exp\left\{-i\sigma\left[(\alpha - \tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{\mu\pi}{2}\right]\right\}$$

where σ is the sign of $\sin\beta$, i.e.,

$$\text{(II-4)} \qquad \sigma = \operatorname{sgn}(\sin\beta) \qquad 0 < \beta < \pi\,, \quad -\pi < \beta \leq 0 \, .$$

From Eq. (II-2) we therefore have:

(II-5) $$P^{\mu}_{\alpha-1}(\cos\theta) \sim \frac{\alpha^{\mu}}{\sqrt{2\pi\alpha\sin\theta_o}} \exp\left\{ -i\sigma \left[\left(\alpha - \frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{\mu\pi}{2} \right] \right\},$$

$$-\pi < \beta < 0, \qquad 0 < \beta < \pi .$$

and from the relations

(II-6) $$\frac{dP_{\alpha-1}(\cos\theta)}{d\theta} = -(\alpha)(\alpha - 1)P^{-1}_{\alpha-1}(\cos\theta), \qquad [[2], \text{p.}63]$$

(II-7) $$\frac{dP^{-1}_{\alpha-1}(\cos\theta)}{d\theta} = P_{\alpha-1}(\cos\theta) - \cos\theta\, P^{-1}_{\alpha-1}(\cos\theta), \qquad [[2], \text{p.}62],$$

it follows that

(II-8) $$\frac{dP_{\alpha-1}(\cos\theta)}{d\theta} \sim \frac{-\alpha}{\sqrt{2\pi\sin\theta\,\alpha}} \exp\left\{ -i\sigma\left[\theta\left(\alpha - \frac{1}{2}\right) - \frac{3\pi}{4}\right]\right\}^{*}$$

(II-9) $$\frac{d^2P_{\alpha-1}(\cos\theta)}{d\theta} \sim \frac{-\alpha}{\sqrt{2\pi\sin\theta\,\alpha}} \exp\left\{ -i\sigma\left[\theta\left(\alpha - \frac{1}{2}\right) - \frac{\pi}{4}\right]\right\},$$

where β is in the ranges $0 < \beta < \pi$, $-\pi < \beta < 0$.

In addition to the above asymptotic expressions the following formulas will be useful below:

From Eq. (3.28) we get

(II-10) $$K^{+}\left(\alpha - \frac{1}{2}, \theta_o\right) \sim k^{+}(\theta, \chi_o)\alpha^{1/2} = k^{+}(\theta, \chi_o)\sqrt{|\alpha|}\exp[i\beta/2], \quad -\pi < \beta < \pi .$$

In addition we have

(II-11) $$\sin\pi\alpha \sim -\frac{\sigma}{2i}\exp(-i\sigma\pi\alpha) \qquad \beta \neq 0, \quad \beta \neq \pi$$

$$r^{-\alpha} = \exp(-\alpha\log r) .$$

---

* Note that $\frac{dP_{\alpha-1}(\cos\theta)}{d\theta}$ is of exponential order unity in α.

Now let g^2 be the integrand of Eq. (4.26'). Clearly

$$g^2 = \frac{r^{-\alpha} P_{\alpha-1}(-\cos\theta) K^+(\nu,\theta_o)}{\alpha P'_{\alpha-1}(\cos\chi_o)} \quad . \tag{II-12}$$

Similarly, let g_r^2, g_{rr}^2, g_θ^2, $g_{\theta\theta}^2$, be the integrands obtained from Eq. (4.26') by differentiating with $\frac{\partial}{\partial r}$, $\frac{\partial^2}{\partial r^2}$, $\frac{\partial}{\partial\theta}$, $\frac{\partial^2}{\partial\theta^2}$, respectively, under the integral sign. Furthermore, let us set*

$$E(\beta) = \sqrt{\frac{\sin\theta_o}{\sin\theta}}\, \exp\left\{ i\sigma \left[(\theta - \theta_o)(\alpha - \tfrac{1}{2}) \right] \right\}, \qquad \beta \neq 0,\ \beta \neq \pi \quad . \tag{II-13}$$

Employing Eqs. (II-5) to (II-12) and the fact that

$$\frac{dP_{\alpha-1}(-\cos\theta)}{d\theta} = -\frac{dP_{\alpha-1}(\cos\chi)}{d\chi}, \qquad \chi = \pi - \theta,$$

the asymptotic behavior of the g's for $\beta \neq 0$ and $\beta \neq \pi$ is found to be the following:

(II-14)

a) $$g^2(r,\theta) \sim \frac{\sigma k^+(\theta_o, \chi_o)}{i\alpha\sqrt{\alpha}}\, r^{-\alpha} E(\beta)$$

b) $$g_r^2(r,\theta) \sim -\frac{\sigma k^+(\theta_o, \chi_o)}{i\sqrt{\alpha}}\, r^{-\alpha-1} E(\beta)$$

c) $$g_{rr}^2(r,\theta) \sim \frac{\sigma k^+(\theta_o, \chi_o)\sqrt{\alpha}}{i}\, r^{-\alpha-2} E(\beta)$$

d) $$g_\theta^2(r,\theta) \sim \frac{k^+(\theta_o, \chi_o)}{\sqrt{\alpha}}\, r^{-\alpha} E(\beta)$$

e) $$g_{\theta\theta}^2(r,\theta) \sim \frac{\sigma k^+(\theta_o, \chi_o)\sqrt{\alpha}}{i}\, r^{-\alpha} E(\beta),$$

---

* We confine our considerations to the integral representation of $U^2(r,\theta)$ which involves θ in the range, $\theta_o \leq \theta \leq \pi$. The corresponding result for U^1 involving θ in the range $0 \leq \theta \leq \theta_o$ can be obtained in a similar manner.

where β is in the range $0 < \beta < \pi$, $-\pi < \beta < 0$. When $\beta = 0$, it follows from Eqs. (II-1) and (II-6) that $E(\beta)$ above must be replaced by a quotient of trigonometric functions. It is furthermore clear that this quotient remains bounded for a suitable choice of $\left\{\alpha_o^n\right\}$ where $|\alpha_o^n|$ approaches infinity through positive real values of α as n approaches infinity. Since

$$\text{(II-15)} \qquad P_{\alpha-1}(\cos\Theta) = P_{-\alpha}(\cos\Theta)$$

a similar statement is valid when $\beta = \pi$. When evaluating the various representations by residues we may therefore employ a sequence of contours through the $|\alpha_o^n|$ [cf. b) p. 22 and f) p. 26].

The following facts are now apparent from an inspection of Eq. (II-14): First, $g(r,\Theta) = O\left\{|\alpha|^{-3/2}\exp\left[-(\Theta-\Theta_o)|\alpha|\right]\right\}$ when α is purely imaginary, that is, when α is on the contour C_ρ [see Fig. 3]. Secondly, for any Θ, $\Theta_o \leq \Theta \leq \pi$ it is easily verified that $|r^{-\alpha}| = e^{-|\alpha|\cos\beta\log r}$ is exponentially damped as $|\alpha| \rightarrow \infty$ when $r > 1$ and β is in the range $\frac{\pi}{2} > \beta > -\frac{\pi}{2}$, or when $r < 1$ and β is in the range $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$. Finally, when $\Theta - \Theta_o = \triangle\Theta$ is greater than zero g and all its derivatives are exponentially damped as $|\alpha|$ becomes large in the angular regions $-\pi < \beta < 0$, $0 < \beta < \pi$.

Employing these results it is a simple matter to verify the statements (a) through (f) on pp. 22 through 26*. To prove the statement (g), let us consider the difference, $\triangle$, of say $\left.\frac{\partial^2 U^1}{\partial\Theta^1}\right|_{\Theta\uparrow\Theta_o}$ and $\left.\frac{\partial^2 U^2}{\partial\Theta^2}\right|_{\Theta\downarrow\Theta_o}$. From Eq. (4.21) and statement (d) we have for $r < 1$

$$\text{(II-16)} \qquad \triangle = \frac{N}{2\pi i}\int_{C^-_{\pi/4}} r^{-\alpha}\left[\frac{P'_{\alpha-1}(\cos\chi_o)P''_{\alpha-1}(\cos\Theta_o) - P'_{\alpha-1}(\cos\Theta_o)P''_{\alpha-1}(\cos\chi_o)}{\alpha\, W(\alpha-\frac{1}{2},\Theta_o)K^-(\alpha-\frac{1}{2},\Theta_o)}\right]d\alpha.$$

But from Legendre's differential equations, namely

---

* Note that $K^+(\nu)K^-(\nu) = \frac{P'_{(2\nu-1)/2}(\cos\Theta_o)P'_{(2\nu-1)/2}(-\cos\Theta_o)}{W(\nu,\Theta_o)}$ [cf. Eqs. (3.18) and (3.19)] and $\nu = \alpha - \frac{1}{2}$, and hence g^2 may also be written as follows:

$$g^2 = \frac{P'_{\alpha-1}(\cos\Theta_o)P(-\cos\Theta_o)}{\alpha\, W(\alpha-\frac{1}{2},\Theta_o)}\, r^{-\alpha}$$

$$\frac{d^2P_{\alpha-1}(\cos\Theta)}{\partial\Theta^2} = \text{ctn}\,\Theta\,\frac{dP_{\alpha-1}(\cos\Theta)}{\partial\Theta} + \alpha(\alpha-1)P_{\alpha-1}(\cos\Theta)$$

(II-17)

$$\frac{d^2P(\cos\chi)}{\partial\chi^2} = \text{ctn}\,\chi\,\frac{dP_{\alpha-1}(\cos\chi)}{\partial\chi} + \alpha(\alpha-1)P_{\alpha-1}(\cos\chi),\quad \chi = \pi - \Theta,$$

and from the fact that

(II-18) $$\frac{dP_{\alpha-1}(\cos\chi)}{\partial\chi} = -\frac{dP_{\alpha-1}(-\cos\Theta)}{\partial\Theta} = -P'_{\alpha-1}(-\cos\Theta),$$

we have

$$\triangle = \frac{N}{2\pi i}\int_{C^-_{\pi/4}} r^{-\alpha}\,\frac{(\alpha-1)}{K^-(\alpha-\frac{1}{2},\Theta_o)}\left[\frac{P'_{\alpha-1}(\cos\chi_o)P_{\alpha-1}(\cos\Theta_o) - P'_{\alpha-1}(\cos\Theta_o)P_{\alpha-1}(\cos\chi_o)}{W(\alpha-\frac{1}{2},\Theta_o)}\right]d\alpha$$

(II-19)

$$= \frac{N}{2\pi i}\int_{C^-_{\pi/4}} r^{-\alpha}\,\frac{(\alpha-1)}{K^-(\alpha-\frac{1}{2},\Theta_o)}\,d\alpha\,, \qquad 1 > r \geq 0\,,$$

since the expression involving the Legendre functions is by definition the Wronskian $W(\alpha - \frac{1}{2}, \Theta_o)$. Now, since the integrand is regular in the left half-plane we have, on evaluating by residues,

(II-20) $$\triangle \equiv 0\,.$$

The continuity of the remaining derivatives mentioned in (g) across $\Theta = \Theta_o$ and $0 \leq r < 1$ either follow automatically or can be obtained by employing a procedure similar to the one just employed; the statement (g) is therefore verified.

Before closing this discussion we shall prove the following relation:

(II-21) $$k^+(\Theta_o, \chi_o) = i/\sqrt{2}$$

From Eqs. (3.18), (3.19) and (3.4) we have, on setting $\nu = \alpha - \frac{1}{2}$,

(II-22) $$K^{+}(\alpha - \tfrac{1}{2})K^{+}(\tfrac{1}{2} - \alpha) = \frac{\pi \sin\theta_o P'_{\alpha-1}(\cos\theta_o)P'_{\alpha-1}(-\cos\theta_o)}{2 \sin \pi\alpha} .$$

Letting $|\alpha|$ become large and employing Eqs. (II-4) and (II-10) we get

$$\left[k^{+}(\theta_o, \chi_o)\right]^2 \sqrt{-\alpha}\ \sqrt{\alpha} = i\sigma\alpha, \qquad \beta \neq 0, \pi.$$

Now

(II-23) $$\sqrt{-\alpha} = \frac{-i\sigma\sqrt{\alpha}}{2} .$$

As a result

(II-24) $$\left[k^{+}(\theta_o, \chi_o)\right]^2 = -1/2.$$

It follows that

$$k^{+}(\theta_o, \chi_o) = \pm i/\sqrt{2} .$$

To determine which sign is the proper one we note that an alternative expression for $k^{+}(\theta_o, \chi_o)$ is, [cf. Eq. (3.29)],

(II-25) $$k^{+}(\theta_o, \chi_o) = \frac{1}{2\pi}\left[\sin\theta_o P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)(\theta_o\chi_o)^{\frac{1}{2}}\right]^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4})L(\theta_o)L(\chi_o),$$

and that the factor $(2\pi)^{-1}\left[\sin\theta_o(\theta_o\chi_o)^{\frac{1}{2}}\right]^{\frac{1}{2}}\Gamma^2(\frac{1}{4})L(\theta_o)L(\chi_o)$ is positive [cf. Eq. (3.24)]. Since $\left[P'_{-\frac{1}{2}}(\cos\theta_o)P'_{-\frac{1}{2}}(\cos\chi_o)\right]^{1/2}$ is positive imaginary [see Eq. (3.43)] it follows that

(II-26) $$k^{+}(\theta_o, \chi_o) = i/\sqrt{2} .$$

Appendix III

Proof of the Relation $K_o^+(\theta_o)K_1^+(\theta_o) = (\sin\theta_o)/2$.

The object of this appendix is to prove the relation

$$K_o^+(\theta_o)K_1^+(\theta_o) = \frac{\sin\theta_o}{2} . \tag{III-1}$$

From Eq. (5.4) of the text we have

$$K_p(\theta_o) = \lim_{\alpha \to \alpha_p(\theta_o)} \left\{ \frac{K^+(\alpha - \frac{1}{2}, \theta_o)}{\alpha} \right\} , \qquad p = 0, 1, 2, \ldots . \tag{III-2}$$

Setting $\nu = \alpha - \frac{1}{2}$, we get

$$\begin{aligned} K_o(\theta_o) &= \lim_{\nu \to \nu_o(\theta_o)} \left\{ \frac{K^+(\nu, \theta_o)}{\nu + \frac{1}{2}} \right\} , \qquad p = 0 , \\ K_1(\theta_o) &= \lim_{\nu \to \nu_1(\theta_o)} \left\{ \frac{K^+(\nu, \theta_o)}{\nu + \frac{1}{2}} \right\} , \qquad p = 1 , \end{aligned} \tag{III-3}$$

where $\nu_o = -\frac{1}{2}$, $\nu_1 = \frac{1}{2}$, since $\alpha_o(\theta_o) = 0$ and $\alpha_1(\theta_o) = 1$. Thus

$$\begin{aligned} K_o(\theta_o) &= \lim_{\nu \to 1/2} \left\{ \frac{K^+(-\nu, \theta_o)}{\frac{1}{2} - \nu} \right\} = \lim_{\nu \to 1/2} \left\{ \frac{K^-(\nu, \theta_o)}{\frac{1}{2} - \nu} \right\} \\ K_1(\theta_o) &= \lim_{\nu \to 1/2} \left\{ \frac{K^+(\nu, \theta_o)}{\frac{1}{2} + \nu} \right\} . \end{aligned} \tag{III-4}$$

As a result

$$K_o(\theta_o)K_1(\theta_c) = \lim_{\nu \to 1/2} \left[\frac{K^+(\nu, \theta_o)K^-(\nu, \theta_o)}{(\nu - \frac{1}{2})(\nu + \frac{1}{2})} \right]$$

(III-5)

$$= \lim_{\nu \to 1/2} \left[\frac{\frac{1}{2} \sin \theta_o \Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu) P'_{(2\nu-1)/2}(-\cos \theta_o) P'_{(2\nu-1)/2}(\cos \theta_o)}{(\frac{1}{2} + \nu)(\frac{1}{2} - \nu)} \right]$$

the last equation being a consequence of Eq. (3.14). Since*

$$P'_{(2\nu-1)/2}(\pm \cos \theta_o) = \mp (\nu - \tfrac{1}{2})(\nu + \tfrac{1}{2}) P^{-1}_{(2\nu-1)/2}(\pm \cos \theta_o)$$

[cf.Eq. (3.6)] and since

$$\left(\tfrac{1}{2} \pm \nu\right) \Gamma(\tfrac{1}{2} \pm \nu) = \Gamma(\tfrac{3}{2} \pm \nu)$$

we have

$$K_o(\theta_o)K_1(\theta_o) = -\tfrac{1}{2} \sin \theta_o \Gamma(1) \Gamma(2) P_o^{-1}(\cos \theta_o) P_o^{-1}(-\cos \theta_o)$$

(III-6)

$$= -\tfrac{1}{2} \sin \theta_o P_o^{-1}(\cos \theta_o) P_o^{-1}(\cos \chi_o) .$$

Now from [2], p. 63,

(III-7) $\qquad P_o^{-1}(\cos \theta) = \mathrm{ctn}(\theta_o/2), \; P_o^{-1}(\cos \chi_o) = \mathrm{ctn}(\chi_o/2) = \tan(\theta_o/2)$.

Consequently,

(III-8) $\qquad K_o^+(\theta_o)K_1^+(\theta_o) = -\sin(\theta_o/2)$,

which is the desired result.

---

* Note that $P'_{(2\nu-1)/2}(-\cos \theta_o) = \frac{d}{d\theta} P_{(2\nu-1)/2}(-\cos \theta)\Big|_{\theta=\theta_c} = -\frac{d}{d\chi} P_{(2\nu-1)/2}(\cos \chi)\Big|_{\chi=\pi-\theta_o}$

where $\chi = \pi - \theta$.

Appendix IV

A Proof of the Validity of the Asymptotic Formulae of Equation (6.15)

In Section 6.1 we showed that the velocities U_r and U_θ, in the r and θ directions were represented as follows:

$$U_r^2 = -\frac{Ne^{-t}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\exp(-t\zeta/\delta)I_r(\zeta/\delta)}{\delta}\,d\zeta,$$

(IV-1)

$$U_\theta^2 = \frac{Ne^{-t}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\exp(-t\zeta/\delta)I_\theta(\zeta/\delta)}{\delta}\,d\zeta,$$

where

a) $$I_r(\alpha) = \frac{P_{\alpha-1}\left[\cos(\theta_o + \Delta\theta_o)\right]}{P'_{\alpha-1}(\cos\theta_o)},$$

(IV-2)

b) $$I_\theta(\alpha) = \frac{P'_{\alpha-1}\left[\cos(\theta_o + \Delta\theta_o)\right]}{\alpha P'_{\alpha-1}(\cos\theta_o)},$$

and where

a) $$t = \delta(\cos\gamma + \tfrac{\delta}{2}) + O(\delta^2), \qquad 0 \le \gamma \le \pi,$$

(IV-3)

b) $$\Delta\theta_o = \delta\sin\gamma + O(\delta^2), \qquad 0 \le \gamma \le \pi.$$

The object of this appendix is to prove that

$$\lim_{\delta\to 0}(\delta^{\frac{1}{2}}U_r^2) = c\left[\int_o^{i\infty}\frac{\exp\left[-\zeta e^{-i\gamma}\right]}{\sqrt{\zeta}}\,d\zeta + \int_o^{-i\infty}\frac{\exp\left[-\zeta e^{i\gamma}\right]}{\sqrt{\zeta}}\,d\zeta\right],$$

(IV-4)

$$\lim_{\delta\to 0}(\delta^{\frac{1}{2}}U_\theta^2) = c\left[\int_o^{i\infty}\frac{\exp\left[-\zeta e^{-i\gamma}\right]}{i\sqrt{\zeta}}\,d\zeta - \int_o^{-i\infty}\frac{\exp\left[-\zeta e^{i\gamma}\right]}{i\sqrt{\zeta}}\,d\zeta\right]$$

uniformly in γ, $0 \le \gamma \le \pi$, where we have written

(IV-5) $$c = (Nk^+(\theta_o, \chi_o)/2\pi).$$

To illustrate the method of proof it is sufficient to show how the first equation of (IV-4) is derived from the first equation of (IV-1). The second equation of (IV-4) follows from the second equation of (IV-1) in a similar manner.

From the first equation of (IV-1), after making the change of variables

(IV-6) $$\zeta = i\,s,$$

we have for the quantity $\delta^{1/2}U$ the following expression:

(IV-7) $$\delta^{1/2}U_r^2 = -\frac{Ne^{-t}}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp(-is/\sin\gamma/r)}{\sqrt{is}}\left\{\sqrt{(is/\delta)}\,\exp\left[-(ist/\delta)+(|s|\sin\gamma/r)\right] I_r(is/\delta)\right\} dis$$

where r is expressed in terms of δ by means of the following relation:

(IV-7') $$r = \sqrt{1 + 2\delta\cos\gamma + \delta^2}\ ,$$

[cf. Eq. (6.9)].

Now from Appendix II, Eqs. (II-13) and (II-14), we have for large $|\alpha|$

(IV-8) $$\sqrt{\alpha}\ I_r(\alpha) \sim i\sigma k^{+}(\theta_o, \chi_o)\ \frac{\sin\theta_o}{\sin(\theta_o + \Delta\theta_o)}\ \exp\left\{i\sigma\,\Delta\theta_o(\alpha - \tfrac{1}{2})\right\}\ ,$$

where

(IV-9) $$\alpha = |\alpha|\ \exp(i\beta), \qquad 0 \le \beta \le 2\pi,$$

and where

(IV-10) $$\sigma = \mathrm{sign}(\sin\beta), \qquad 0 < \beta\ \rho\ \pi,\ \pi < \beta < 2\pi.$$

Our first objective is to prove that the absolute value of the braced term, say $B(s,\delta)$, in the integrand of Eq. (IV-7) is bounded as $|s/\delta|$ approaches infinity. To this end we substitute $\alpha = is/\delta$ in Eq. (IV-8) and verify that

$|B(s,\delta)|$ is bounded as $|s/\delta|$ approaches infinity provided the same is true of the expression $\exp\left[-\frac{|s|}{\delta}\left(\triangle\theta_o - \frac{\delta\sin\gamma}{r}\right)\right]$. Now $\triangle\theta_o = w/r$ where w is the arc length subtended by $\triangle\theta_o$ on the circle r = const. From Fig. 5 it is clear that $\delta\sin\gamma$ is one half the length of the chord joining the two points of intersection of the circle determined by r = const and δ = const. Clearly then $\triangle\theta_o-(\delta\sin\gamma/r) \geq 0$, the minimum of this function occuring at $\gamma = 0$ or $\gamma = \pi$. It follows that

$$\lim_{|s/\delta|\to\infty} |B(s,\delta)| \leq \max_{s,\delta}\left\{\exp\left[-\frac{|s|}{\delta}\left(\triangle\theta_o - \frac{\delta\sin\gamma}{r}\right)\right]\right\} \leq 1.$$

In addition since $\sqrt{\alpha}\, I(\alpha)$ is regular for all non-real α it is a priori continuous on the imaginary α-axis. It follows that $|\sqrt{is/\delta}\exp(|s|\sin\gamma)I(is/\delta)|$ is uniformly bounded for all s, the bound, which we denote by M, not depending on δ, but perhaps on γ. For any $\varepsilon_1 > 0$ sufficiently small and for any γ such that $\varepsilon_1 \leq \gamma \leq \pi-\varepsilon_1$ it is clear, therefore, that the integral in (IV-7) is absolutely convergent for all $\delta > 0$, the absolute value of the integrand being dominated by the integrable function $(M\exp(-s\sin\gamma)/\sqrt{s})$. Employing the dominated convergence theorem we have for $\varepsilon_1 \leq \gamma \leq \pi-\varepsilon_1$

$$\text{(IV-11)}\quad \lim_{\delta\to 0}(\delta^{\frac{1}{2}}U_r^2) = -\frac{N}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp(-is\cos\gamma-|s|\sin\gamma)}{\sqrt{is}}\lim_{\delta\to 0}\left[\sqrt{is/\delta}\,\exp(|s|\sin\gamma I_r(is/\delta)\right]d(is)$$

$$= -\frac{N\kappa^+(\theta_o\chi_o)}{(2\pi i^2)}\left[\int_0^{\infty}\frac{\exp(-is\cos\gamma-s\sin\gamma)}{\sqrt{is}}\,d(is)\right.$$

$$\left. -\int_{\infty}^{0}\frac{\exp(-is\cos\gamma+s\sin\gamma)}{\sqrt{is}}\,d(is)\right],$$

where the first equation follows from (IV-3a) and the second from (IV-3b) and (IV-8) - (IV-10). Setting $s = \zeta/i$ we find, using Eq. (IV-5), that (IV-4) is

satisfied for $\epsilon_1 \leq \gamma_1 \leq \pi-\epsilon_1$. To extend the result to the entire interval $0 \leq \gamma \leq \pi$ it is first necessary to deform the integrals in (IV-1) to the contours $C^+_{\pi/4}$ when $0 \leq \gamma < \pi/2$ and to $C^-_{\pi/4}$ when $\pi/2 < \gamma \leq \pi$. Employing a procedure similar, except for minor modifications, to the one employed above we have

$$\text{(IV-12a)} \quad \lim_{\delta \to 0} (\delta^{\frac{1}{2}} U_r^2) = c \left[\int_0^{\infty \exp(i\pi/4)} \frac{\exp\left[-\zeta e^{-i\gamma}\right]}{\sqrt{\zeta}} d\zeta + \int_0^{\infty \exp(i7\pi/4)} \frac{\exp\left[-\zeta e^{i\gamma}\right]}{\sqrt{\zeta}} d\zeta \right]$$

uniformly for all γ, $0 \leq \gamma \leq (\pi/2)-\epsilon_2$ where $\epsilon_2 > 0$, and in addition we find that

$$\text{(IV-12b)} \quad \lim_{\delta \to 0} (\delta^{\frac{1}{2}} U_r^2) = c \left[\int_0^{\infty \exp(i3\pi/4)} \frac{\exp\left[-\zeta e^{-i\gamma}\right]}{\sqrt{\zeta}} d\zeta + \int_0^{\infty \exp(i5\pi/4)} \frac{\exp\left[-\zeta e^{i\gamma}\right]}{\sqrt{\zeta}} d\zeta \right]$$

uniformly for all γ, $\pi \leq \gamma \leq \pi/2 + \epsilon_3$, $\epsilon_3 > 0$. By deforming the paths of integration of the first integrals in (IV-12a) and (IV-12b) into the positive imaginary axis and the paths of the second integrals into the negative imaginary axis we obtain expressions for $\lim(\delta^{1/2} U_r^2)$ which are identical with that given in (IV-4). It follows that $\delta^{1/2} U_r$ converges to the expression given in the right-hand side of the first equation of (IV-4) for $0 \leq \gamma \leq \pi$ and the convergence is uniform for γ in this range.

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